Fundamental notions of graph theory

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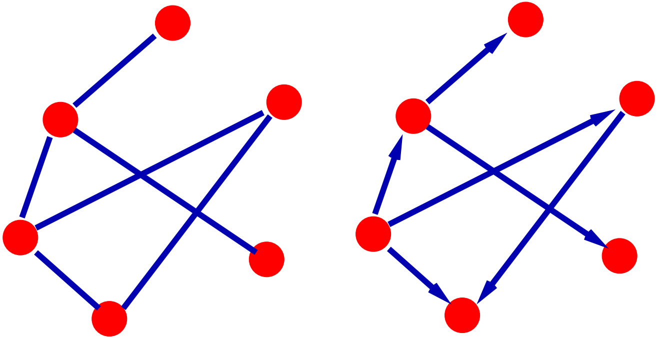
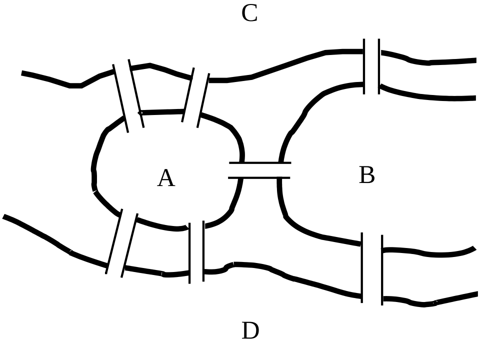
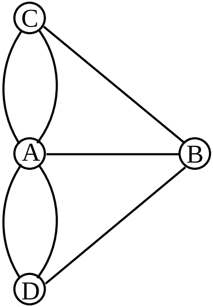
Graphs and digraphs



Problem of 7 bridhes in Kaliningrad

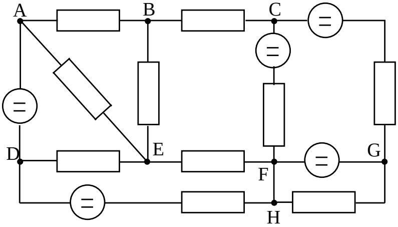
Problem of 7 bridges in Kaliningrad – Köningsberg
Leonhard Euler – 1736

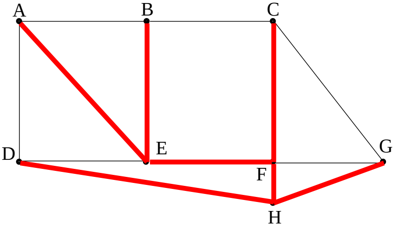




Solving of electric circuits

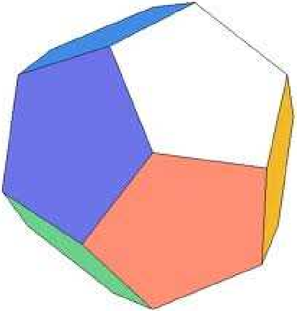
R. 1847 Kirchoff proposed how to solve complex electric circiuts using its partial sub scheme which is in contemporary graph terminology called as spanning tree.





Hamiltonian paths

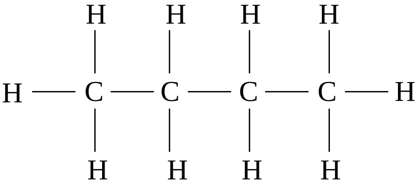
Irish mathematian R. W. Hamilton r. 1859 studied problems how to travel along edges and vertices of dodecahedron in order to visit every vertex exactly one time. This problem is a predecessor of famous problem called travelling salesman problem.

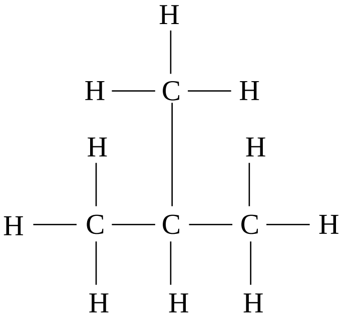


Chemical graphs

1874 – Caley studied structure types of chemical formulas. He used a graphical pictures similar to todays graphs.

Sylvester in 1878 used in this content term graph for the first time.





Origin of moder graph theory

- 1936 – Hungarian mathematician D. König published the first monography about graph theory.
- 1975 Christofides published the first monography whole devoted to algorithmic graph theory
- 1983 – J. Plesník published Slovak book *Grafové algoritmy*

In 1965 Edmonds was the first to realize that there are two types of algorithms

- good – having polynomial complexity
- bad – comlexity of which can not be limited from above with and polynomial

A new mathematical discipline originated. This discipline studies complexity of algorithms and complexity of problems.

Definition of a graph

Definition

An ordered couple (u, v) *of elements* u, v *of the set* V *is such couple, where it is determined which of elements* u, v *is in the first and which is in the second place.*
An ordered n**-tuple** *of elements is such a* n*-tuple of elements* $(a_1, a_2, \dots, a_n)$, *where the order of elements is determined.*

Definition

A graph *is an ordered couple of sets* $G = (V, H)$, *where* V *is a nonempty finite set and* H *is a set of non ordered couples of the type* $\{u, v\}$ *such that* $u \in V$, $v \in V$ *and* $u \neq v$, *i. e.*

$$H \subseteq \{\{u, v\} \mid u \neq v,\ u, v \in V\}. \tag{1}$$

Elements of the set V *are called* **vertices** *and elements of the set* H *are called* **edges** *of the graph* G.

Definition of a digraph

Definition

A digraph *is an ordered couple of sets* $\overrightarrow{G} = (V, H)$*, where* V *is a nonempty finite set and* H *is a set of ordered couples of the type* (u, v) *such that* $u \in V$, $v \in V$ *and* $u \neq v$*, i.e.*

$$H \subseteq \{(u, v) \mid u \neq v,\ u, v \in V\}. \tag{2}$$

Elements of the set V *are called* **vertices** *and elements of the set* H *are called* **directed edges** *or* **arcs** *of the digraph* $\overrightarrow{G}$*.*

- There is enormous disorder in graph terminology
- Vertices are sometimes called as nodes
- Directed edges are sometimes called as arcs

Digraph – a set V with antireflexive relation

Graph – a set V with antireflexive symmetric relation

Diagram of a graph

Definition

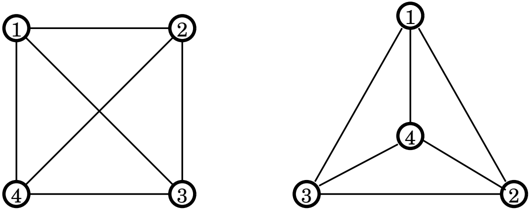
Diagram of a graph. *Graph is often represented by a picture called diagram of graph.* **A diagram of a graph** $G = (V, H)$ *in some space* $\mathcal{P}$ *is a set* B *of points and a set* S *of continuos (curved) lines in space* $\mathcal{P}$ *such that*

- *Exactly one point* $b_v \in B$ *corresponds to exactly one vertex* $v \in V$ *such that if* $u, v \in V,\ u \neq v$ *then* $b_u \neq b_v$.
- *Exactly one line* $s \in S$ *corresponds to exactly one edge* $h \in H$ *such that if* $h, k \in H,\ h \neq k$ *then* $s_h \neq s_k$.
- *If* $h = \{u, v\} \in H$, *then the line* s_h *has end points* b_u, b_v. *None edge contains any point from* B *exepting end points.*
- *Moreover, two different lines can intersect at most once and no line can intersect itself.*

Planar diagram, planar graph

Definition

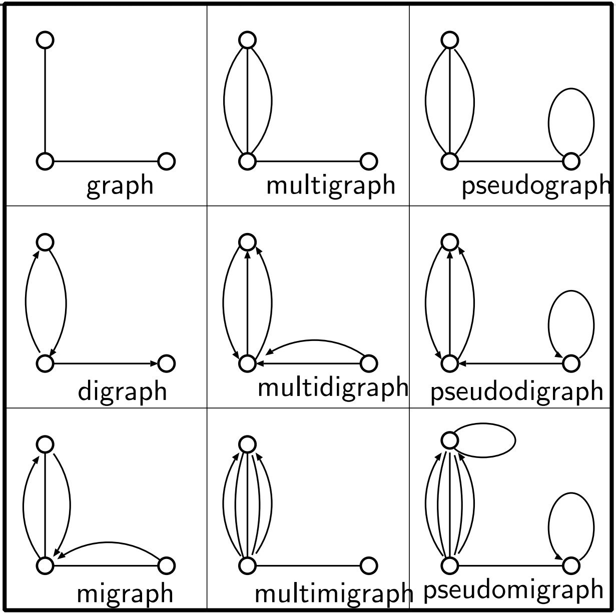
Diagram of a graph or a digraph in a plane is called **planar** *if its edges do not intersect one another nowhere else but in vertices of diagram.*
Graph $G = (V, H)$, resp. digraph $\overrightarrow{G} = (V, H)$ ic called **planar**, *if there exists its planar diagram.*



Obr.: Two diagrams of the same graph $G = (V, H)$,

kde $V = \{1, 2, 3, 4\}$, H=$\{\{1, 2\}$,$\{1, 3\}$,$\{1, 4\}$, $\{2, 3\}$,$\{2, 4\}$,$\{3, 4\}\}$.

More general graph structures



Subgraph

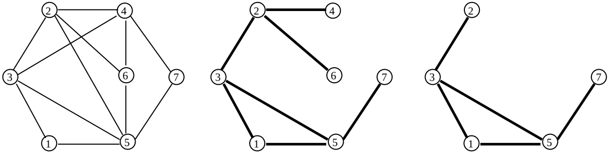
Definition

A graph $G' = (V', H')$ *is* **a subgraph of a graph** $G = (V, H)$*, if it holds* $V' \subseteq V$ *a* $H' \subseteq H$*. In this case we will write* $G' \subseteq G$*.*
A digraph $\overrightarrow{G'} = (V', H')$ *is* **a subgraph of a digraph** $\overrightarrow{G} = (V, H)$*, if* $V' \subseteq V$ *and* $H' \subseteq H$*.* [a]

[a] G' read as „*G dashed*“

Definition

We will say that a graph graph $G' = (V', H')$ *is* **a spanning subgraph** *of a graph* $G = (V, H)$*, if it holds* $V' = V$ *and* $H' \subseteq H$*. Similarly we define* **a spanning subdigraph of a digraph** $\overrightarrow{G}$*.*



Example

Let $G = (V, H)$ is a graph. If it holds $V' \subseteq V$, $H' \subseteq H$ for a couple $G' = (V', H')$ G' still need not be a subgraph of G.

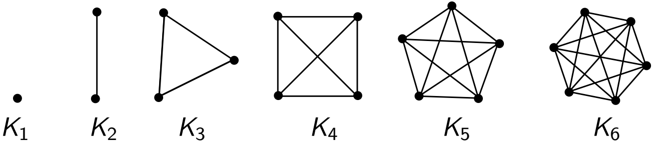
Example

$G = (\{1,2,3\}, \{\{1,2\},\{1,3\}\})$, $G' = (\{1,2\}, \{\{1,3\}\})$.
G' is not a graph because the edge $\{1,3\}$ is not a couple of elements from $\{1,2\}$.

Complete graph

Definition

A graph $G = (V, H)$ is called **complete**, *if the set H contains all possible couples of the type $\{u, v\}$ where $u, v \in V$ and $u \neq v$. A complete graph with n vertices wil be denoted by K_n.*
A digraph $\overrightarrow{G} = (V, H)$ is called **complete**, *if the set H contains all possible ordered couples of the type (u, v) where $u, v \in V$ and $u \neq v$.*



Obr.: Diagrams of complete graphs K_1 až K_6.

Maximum resp. minimum subgraph of a graph with property $\mathcal{V}$

Definition

A maximum subgraph G' of a graph G with a property $\mathcal{V}$ *is such a subgraph of G which*

1. *has the propery $\mathcal{V}$,*
2. *there does not exists a subgraph G'' of G with property $\mathcal{V}$ such that $G' \subseteq G''$ and $G' \neq G''$.* [a]

A minimum subgraph G' of a graph G with property $\mathcal{V}$ *is such a subgraph of G which*

1. *has the propery $\mathcal{V}$,*
2. *there does not exists a subgraph G'' of G with property $\mathcal{V}$ such that $G'' \subseteq G'$ and $G' \neq G''$.*

[a] G' read as „G dashed“, G'' read as „G double dashed“

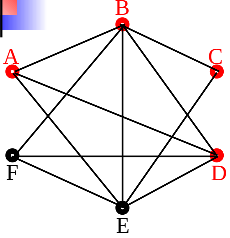
Subgraphs induced by a set of vertices resp. by a set of edges

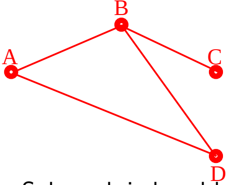
Definition

Let $G = (V, H)$ is a graph (digraph), $V' \subseteq V$. We will say that G' is **a subgraph of a graph (digraph)** G **induced by the set of vertices** V'*, if G' is a maximum subgraph of G having V' as the set of vertices.*

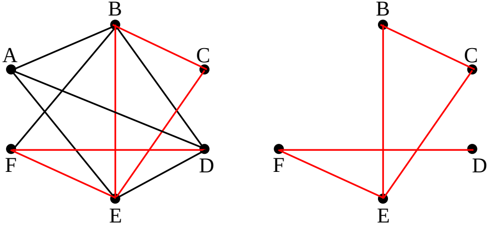
Let $H' \subseteq H$. We will say that G' is **a subgraph of a graph (digraph)** G **induced by the set of edges** H'*, if G' is a minimum subgraph of G having H' as the set of edges.*

Subgraphs induced by a set of vertices resp. by a set of edges





Subgraph induced by the set of vertices $\{A, B, C, D\}$



Subgraph induced by the set of edges $\{\{B,C\},\{B,E\},\{C,E\}\{E,F\},\{F,D\}\}$

Adjacency and incidence

Definition

Let $G = (V, H)$ be a graph, resp. digraph, $v \in V$, $h \in H$.
Vertex v is **incident with edge** *h, if v is one of vertices of the edge h.*
Edges $h,\ k \in H$, $h \neq k$ are **adjacent***, if they have one vertex in common.*
Vertices u, v are **adjacent***, if $\{u, v\} \in H$, i.e. if $\{u, v\}$ is an edge, (resp. if $(u, v) \in H$ or $(v, u) \in H$).*
Vertices u, v are called **endpoints of the edge** *$h = \{u, v\} \in H$.*

Remark

Vertices u, v are adjacent, if they are "joined" with an edge.

We will denote by $H(v)$ the set of all edges of graph G incident with vertex v.

We will denote by $V(v)$ the set of all vertices adjacent to vertex v.

Outgoing and incoming edges

Definition

Let $\overrightarrow{G} = (V, H)$ *be a digraph,* $u \in V$, $v \in V$, $h = (u, v) \in H$. *We will say that* **the directed edge** h **is outgoing from vertex** u *and* **the directed edge** h **is incoming into vertex** u.
Vertex u *is called* **head of** h *and vertex* v *is called* **tail of** h .
Vertices u, v *are called* **endpoints of the edge** $h = (u, v)$.

$H^{+}(v)$ – the set of all arcs outgoing from vertex v

$H^{-}(v)$ – the set of all arcs incoming to vertex v

$V^{+}(v)$ – the set of all tails of all arcs outgoing from vertex v

$V^{-}(v)$ – the set of all heads of all arcs incoming to vertex v

$$H(v) = H^{+}(v) \cup H^{-}(v) \qquad V(v) = V^{+}(v) \cup V^{-}(v)$$

Neighborhood

Definition

Let $G = (V, H)$ is a graph or digraph, $v \in V$.

Neighborhood of the vertex *v is a graph, (resp. digraph)*

$$O(v) = (V(v) \cup \{v\}, H(v)),$$

i. e. a graph, vertex set of which consist from vertex v and from all vertices adjacent with v and edge set of which is the set of all edges incident with vertex v.

Forward star, backward star

Definition

Let $\overrightarrow{G} = (V, H)$ is a digraph, $v \in V$.

Forward start of the vertex *v is a digraph*

$$Fstar(v) = (V^{+}(v) \cup \{v\}, H^{+}(v)),$$

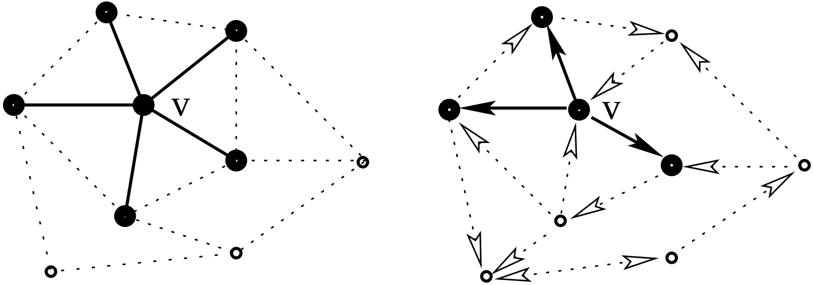
i.e. a digraph, vertex set of which consists from vertex v and from all tails of all arcs outgoing from vertex v and edge set of which is the set of all arcs outgiong from vertex v.

Backward star of the vertex *v is a digraph*

$$Bstar(v) = (V^{-}(v) \cup \{v\}, H^{-}(v)),$$

i.e. a digraph, vertex set of which consists from vertex v and from all heads of all arcs incoming to vertex v and edge set of which is the set of all arcs incoming to vertex v.

Neighborhood of a vertex



Neighborhood of vertex *v*

Forward star of vertex *v*

Obr.: Neighborhood and forward star of vertex *v* are drawn with thick lines.

Degree of a vertex

Definition

Degree $\deg(v)$ **of a vertex** $v \in V$ *in a graph* $G = (V, H)$ *is the number of edges incident with vertex* v.

Output degree odeg(v) **of a vertex** $v \in V$ *in a digraph* $\overrightarrow{G} = (V, H)$ *is the number of arcs of* $\overrightarrow{G}$ *outgoing from vertex* v.

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Theorem

(Euler.) *The sum of the degrees of all vertices in any graph* $G = (V, H)$ *is twice the number of edges, i.e.*

$$\sum_{v \in V} \deg(v) = 2.|H|.$$

The number of vertices with odd degree is even

Theorem

The number of all vertices with odd degree in arbitrary graph $G = (V, H)$ is even.

$$V = V_1 \cup V_2$$

- V_1 – the set of all vertices with odd degree
- V_2 – the set of all vertices with even degree

$$\sum_{v \in V} \deg(v) = \sum_{v \in V_1 \cup V_2} \deg(v) = \sum_{v \in V_1} \deg(v) + \sum_{v \in V_2} \deg(v) = 2.|H|, \quad (3)$$

therefore

$$\sum_{v \in V_1} \deg(v) = 2.|H| - \sum_{v \in V_2} \deg(v). \quad (4)$$

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$$\sum_{v\in V_1} \deg(v) = \underbrace{\underbrace{2.|H|}_{\text{even}} - \sum_{v\in V_2} \underbrace{\deg(v)}_{\text{even}}}_{even} \quad . \tag{5}$$

Colorary: $\sum_{v\in V_1} \deg(v)$ is an even number.

$\sum_{v\in V_1} \deg(v)$ is the sum of k odd numbers.
Let $V_1 = \{v_1, v_2, \dots, v_k\}$, let k is a odd number.
Then

$$\sum_{v\in V_1} \deg(v) = \underbrace{(\deg(v_1) + \deg(v_2))}_{\text{even}} + \underbrace{(\deg(v_3) + \deg(v_4))}_{\text{even}} + \dots$$
$$\dots + \underbrace{(\deg(v_{k-2}) + \deg(v_{k-1}))}_{\text{even}} + \underbrace{\deg(v_k)}_{\text{odd}} \tag{6}$$

If k is odd then $\sum_{v\in V_1} \deg(v)$ is odd, what is in contradiction with (5).

The number of vertices with odd degree is even

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Regular graph, complementary graphs

Definition

A regular graph *is a graph whose vertices all have equal degree.*
A k**-regular graph** *is a regular graph whose common degree is* k*.*

Definition

The **complement of graph** $G = (V, H)$ *is graph* $\overline{G} = (\overline{V}, \overline{H})$ *with the vertex set* $\overline{V}$*,* $V = \overline{V}$ *and with the edge set* $\overline{H}$ *for which it holds:*

$$\{u, v\} \in H \text{ if and only if } \{u, v\} \notin \overline{H}.$$

[a].

Complement of a digraph is defined by the same way.

Let $\overline{G}$ *be a complement of* G*. We will say that* G *and* $\overline{G}$ *are* **complementary graphs***.*

[a] $\overline{G}$ read as „G bar“

Obr.: Pair of complementary graphs and pair of complementary digraphs.

Regular graph, complementary graphs

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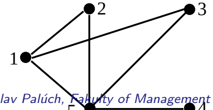
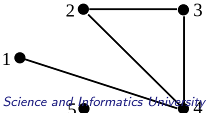
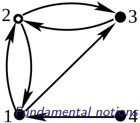
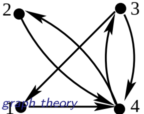




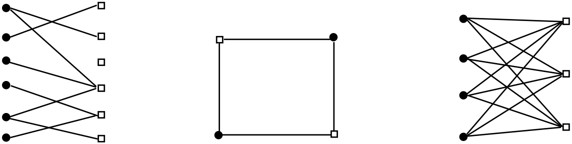




Bipartite graph

Definition

A bipartite graph *is such a graph $G = (V, H)$, whose vertex set V is union of two disjoint sets – parts $V_1,\ V_2$ so that no two vertices from the same part are adjacent.*
A complete bipartite graph *K_{mn} is a bipartite graph with parts V_1, V_2 in which $|V_1| = m$, $|V_2| = n$ and in which every vertex from V_1 is adjacent to ev ery vertx from V_2.*



Obr.: Diagrams of bipartite graphs.

Vertices of parts V_1 are illustrated by balls,
vertices of V_2 are represented by squares.
Diagram in the middle represents bipartite graph $K_{2,2}$,
the third diagram from the left is a diagram of bipartite graph $K_{4,3}$.

Edge graph

Definition

Let $G = (V, H)$ be a graph. **An edge graph** *of G is a graph $L(G) = (H, E)$, whose vertex set is the edge set of G anad whose edge set E is defined as follows: $\{h_1, h_2\} \in E$ if and only if the edges $h_1,\ h_2$ are adjacent.*

Edge weighted graphs

Definition

A graph, resp. digraph $G = (V, H)$ is **edge weighted**, *if a real function $c : H \to \mathbb{R}$ assigning a real number $c(h)$ to every edge $h \in H$ is defined. Real number $c(h)$ assigned to an edge $h \in H$ is called* **weight** *or* **cost** *of the edge h.*

A edge weighted graph can be considered as an ordered triple $G = (V, H, c)$, where (V, H) is a graph with vertex set V, edge set H and where $c : H \to \mathbb{R}$ is a real function defined on the set H.

A vertex weighted graph (digraph) *can be defined by similar way as an ordered triple $G = (V, H, d)$, where (V, H) is a graph with vertex set V, edge set H and where $d : V \to \mathbb{R}$ is a real function defined on the set V. he number $d(v)$ is called* **weight** *or* **cost** *of the vertex v.*

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Isomorphism of graphs and digraphs

Definition

Graph $G = (V, H)$ is **isomorphic with graph** *$G' = (V', H')$, if there exists a one to one mapping (bijection) $f : V \leftrightarrow V'$, such that for every two vertices $u, v \in V$ it holds:*

$$\{u, v\} \in H \quad \textit{if nad only if} \quad \{f(u), f(v)\} \in H'. \tag{7}$$

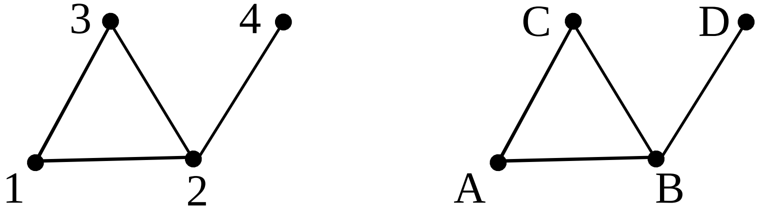
The mapping f is called **isomorphism of graphs** *G and G'.*

Digraph $\overrightarrow{G} = (V, H)$ is **isomorphic with digraph** *$\overrightarrow{G}' = (V', H')$, if there exists an one to one mapping (bijection) $f : V \leftrightarrow V'$, such that for every two vertices $u, v \in V$ it holds:*

$$(u, v) \in H \quad \textit{if and only if} \quad (f(u), f(v)) \in H'. \tag{8}$$

The mapping f is called **isomorphism of digraphs** *$\overrightarrow{G}$ and $\overrightarrow{G}'$.*

An example of isomorphic graphs



Obr.: A pair of isomorphic graphs.

Mapping f defined by ewquations:
$f(1) = A$, $f(2) = B$, $f(3) = C$, $f(4) = D$ is isomorphism.

Remark

Isomorphism of graphs is a reflexive, symmetric and transitive binary relation – isomorphism is an equivalence relation on the class of all graphs.

Invariants of isomorphism

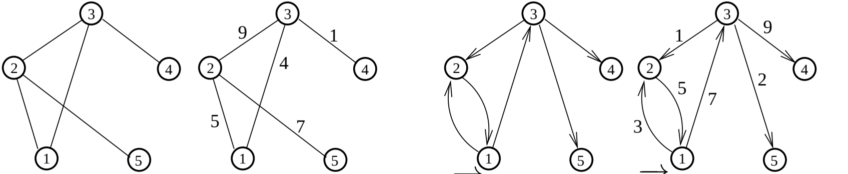
Two isomorphic graphs G, G' have all graph characteristics equal – e.g. number of vertices, number of edges, degrees of vertices, number of components, number of bridges and articulations, etc.
Such characteristics are called **invariants of isomorphism**.
Invariants of isomorphism can be applied to prove that two graphs G, G' are not isomorphic – if graph G has a property different from G', then such two graphs cannot be isomorphic.

To prove that two graphs are isomorphic it is necessary to design a concrete mapping f fulfilling (7), resp. (8).
The only known way in present days how to do it is to try all bijections of $f : V \to V'$. The number of such mappings is $n!$ (where $n = |V|$).

The graph isomorphism problem is to design a general algorithm which would decide whether two graphs are or are not isomorphic, or to prove that such algorithm does not exist.

Representation of graphs and digraphs

1. Representation by diagram of graph



$G_1 = (V_1, H_1)$ $G_1' = (V_1, H_1, c_1)$ $\overrightarrow{G_2} = (V_2, H_2)$ $\overrightarrow{G_2'} = (V_2, H_2, c_2)$

Obr.: Diagrams of a graph and of an edge weighted graph,

digraph and of an edge weighted digraph.

2. Reprezentation by vertex set and edge set

Let $V_1 = \{1, 2, 3, 4, 5\}$, $H_1 = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 5\}, \{3, 4\}\}$. Graph $G_1 = (V_1, H_1)$ is uniquely defined by two sets V_1 a H_1.

Similarly: Leth $V_2 = \{1, 2, 3, 4, 5\}$ and
$H_2 = \{(1, 2), (1, 3), (2, 1), (3, 2), (3, 4), (3, 5)\}$,
the two sets V_2, H_2 uniquely define digraph $\overrightarrow{G_2} = (V_2, H_2)$.

Vertex set V can be represnted in a computer by an one dimensional array $V[\,]$ with $n = |V|$ elements where $V[i]$ is the i-th vertex.

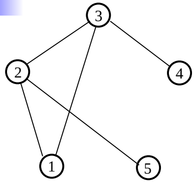
Edge set can be stored in a two dimensional array $H[\,][\,]$ of the type $(m \times 2)$ $(m = |H|\ )$ where $H[j][1]$, $H[j][2]$ are endpoints of j-th edge. This way allows us to store also directed edges – $H[j][1]$ as the head and $H[j][2]$ as the tail of directed edge j.

If we store an edge weighted graph, we can store the weight of an edge j in paralell one dimensional array $C[\,]$ as $C[j]$, or to use two dimensional array $H[\,][\,]$ fo the type m where $H[j][3]$ is the weight of the edge j.

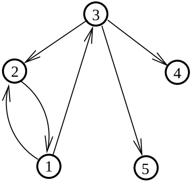
Remark

This way is suitable for multigraphs and miltidigraphs, too.

Example



$G_1 = (V_1, H_1)$



$\overrightarrow{G_2} = (V_2, H_2)$

i	1	2	3	4	5
$V[i]$	1	2	3	4	5

j	1	2	3	4	5
$H[j,1]$	1	1	2	2	3
$H[j,2]$	2	3	3	5	4
$C[j] = H[j,3]$	5	4	9	7	1

i	1	2	3	4	5
$V[i]$	1	2	3	4	5

j	1	2	3	4	5	6
$H[j,1]$	1	1	2	3	3	3
$H[j,2]$	2	3	1	2	4	5
$C[j] = H[j,3]$	3	7	5	1	9	2

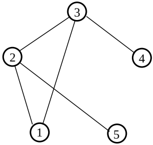
Tabuľka:

Representation of graph G_1 and digraph $\overrightarrow{G_2}$.

3. Representation by adjacency matrix

An adjacency matrix M $= (m_{ij})$ is a square matrix of the type $n \times n$, where $n = |V|$ is the nuber of vertices of graph, resp. digraph G. Elements of adjacency matrix are defined as follows:

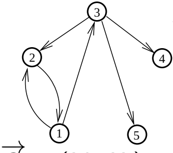
$$m_{ij} = \begin{cases} 1 & \text{if } \{i,j\} \in H \\ 0 & \text{otherwise} \end{cases} \qquad m_{ij} = \begin{cases} 1 & \text{if } (i,j) \in H \\ 0 & \text{otherwise} \end{cases} \tag{9}$$



$G_1 = (V_1, H_1)$

	1	2	3	4	5
1	-	1	1	-	-
2	1	-	1	-	1
3	1	1	-	1	-
4	-	-	1	-	-
5	-	1	-	-	-

Adjacency matrix of graph G_1.



$\overrightarrow{G_2} = (V_2, H_2)$

	1	2	3	4	5
1	-	1	1	-	-
2	1	-	-	-	-
3	-	1	-	1	1
4	-	-	-	-	-
5	-	-	-	-	-

Adjacency matrix of digraph $\overrightarrow{G}_2$.

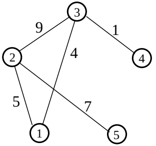
Remark

Adjacency matrix can not be used for multigraphs.

4. Representation by edge weigth matrix

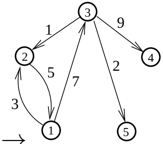
An edge weight matrix M $= (m_{ij})$ of a graph G is a square matrix of the type $n \times n$, where $n = |V|$ is the nmuber of vertices of graph, resp. digraph G. Elements of adjacency matrix are defined as follows:

$$m_{ij} = \begin{cases} c(\{i,j\}) & \text{if } \{i,j\} \in H \\ \infty & \text{otherwise} \end{cases} \qquad m_{ij} = \begin{cases} c((i,j)) & \text{if } (i,j) \in H \\ \infty & \text{otherwise} \end{cases} \tag{10}$$



$G'_1 = (V_1, H_1, c_1)$

	1	2	3	4	5
1	-	5	4	-	-
2	5	-	9	-	7
3	4	9	-	1	-
4	-	-	1	-	-
5	-	7	-	-	-



$\overrightarrow{G'_2} = (V_2, H_2, c_2)$

	1	2	3	4	5
1	-	3	7	-	-
2	5	-	-	-	-
3	-	1	-	9	2
4	-	-	-	-	-
5	-	-	-	-	-

Remark

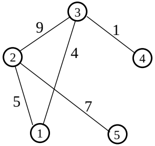
Edge weight matrix can not be used for edge weighted multigraphs.

5. Representation by list of neighbors

Every graph can be stored in such a way that we cite the list of neighbors $V(v)$ for every vertex v.

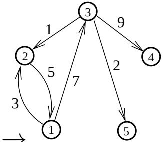
Similarly a digraph can be represented in such a way that we specify the set $V^+(v)$ of all tails of arcs outgoing from every vertx v.

Both mentioned ways for graph G_1 and digraph $\overrightarrow{G}_2$ are apparently visible on following tables:



$G_1' = (V_1, H_1, c_1)$

$V(1)$	2	3	-
$V(2)$	1	3	5
$V(3)$	1	2	4
$V(4)$	3	-	-
$V(5)$	2	-	-



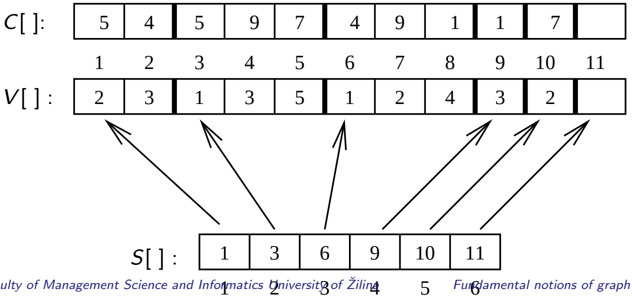
$\overrightarrow{G_2'} = (V_2, H_2, c_2)$

$V^+(1)$	2	3	-
$V^+(2)$	1	-	-
$V^+(3)$	2	4	5
$V^+(4)$	-	-	-
$V^+(5)$	-	-	-

An Example of Effective Storage

Lists of neighbors resp. lists of tails of outgiong arcs can be implemented very effectively in the following way:

1. Create one dimensional array $V[\,]$.
2. Store into array $V[\,]$ all neighbors of vertex 1, then all neighbors of vertex 2, e.t.c., then all neighbors of last vertex n.
3. Create one dimensional array $S[\,]$.
4. For every vertex i set $S[i]$ equal to the first index k in array $V[\,]$ where $V[k]$ is a neighor of i.
5. Set $S[n+1]$ (where $n = |V|$) equal to the first unused place in $V[\,]$.
6. For edge weighted graph store adequat edge weightes in array $C[\,]$ parallel with array $V[\,]$.



	1	2	3	4	5	6	7	8	9	10	11
$C[\,]$:	5	4	5	9	7	4	9	1	1	7	
$V[\,]$:	2	3	1	3	5	1	2	4	3	2	

	1	2	3	4	5	6
$S[\,]$:	1	3	6	9	10	11

6. Representation by Incidence Matrix

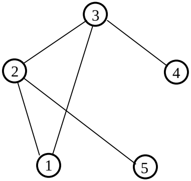
An incidence matrix is a matrix $\mathbf{B}$ of the type $n \times m$, where $n = |V|$ is the number of vertices and $m = |H|$ is the number of edges of represented graph.
Every element b_{ij} of matrix $\mathbf{B}$ says about a way of incidence of vertex i with edge j as folows:

$$b_{ij} = \begin{cases} 1 & \text{ak vertex } i \text{ is incident with edge } j \text{ in graph } G \\ 0 & \text{otherwise} \end{cases}$$

$$b_{ij} = \begin{cases} 1 & \text{if vertx } i \text{ is head of arc } j \text{ in digraph } \overrightarrow{G} \\ -1 & \text{if vertex } i \text{ is tail of arc } j \text{ in digraph } \overrightarrow{G} \\ 0 & \text{otherwise} \end{cases}$$

This way is suitable for multigraphs and multidigraphs
It is possible to define $b_{ii} = 2$ for loops $\{i, i\}$ in pseudographs and $b_{ii} = -2$ for directed loops (i, i) in pseudodigraphs.

Example



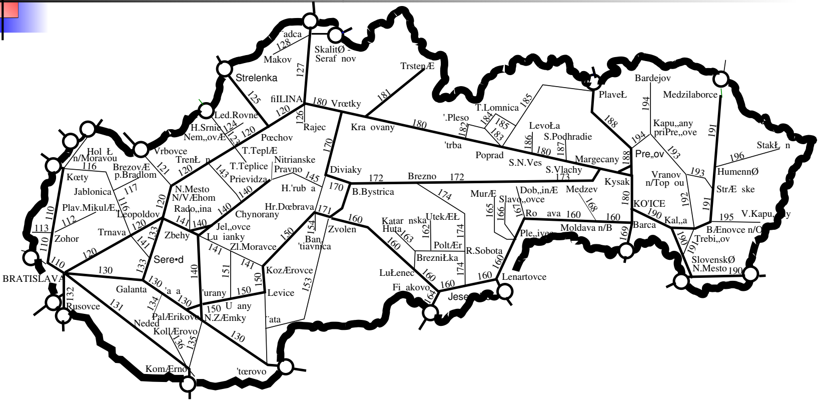
$G_1 = (V_1, H_1)$

v	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{2,5\}$	$\{3,4\}$
1	1	1			
2	1		1	1	
3		1	1		1
4					1
5				1	

Tabuľka: Incidence matrix of graph $G_1 = (V_1, H_1)$

$(V_1 = \{1,2,3,4,5\},\ H_1 = \{\{1,2\},\{1,3\},\{2,3\},\{2,5\},\{3,4\}\})$.

Application – Model of Slovak Railway Network



Obr.: Model of Slovak railway network.

This picture can be considered as a diagram of an edge weighted graph G. Edge weights represents belonginng of corresponding modeled segment to railway line and are used to easy determining of trips traveling along corresponding edge.