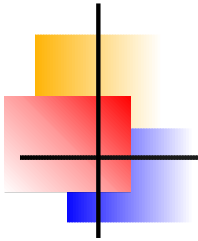

Paths in Graphs

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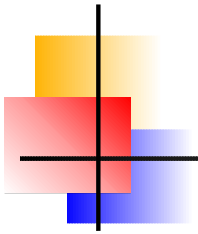
Fakulta riadenia a informatiky, Žilinská univerzita

18. mája 2016



Walk, trail, path

Definition

Let $G = (V, H)$ be a graph.

A walk (a v_1–v_k walk) *in graph G is arbitrary alternating sequence of vertices and edges in the form:*

$$\mu(v_1, v_k) = (v_1, \{v_1, v_2\}, v_2, \{v_2, v_3\}, v_3, \ldots, \{v_{k-1}, v_k\}, v_k). \qquad (1)$$

A trail (a v_1–v_k trail) *in graph G is a v_1–v_k walk in graph G with no repeated edges.*

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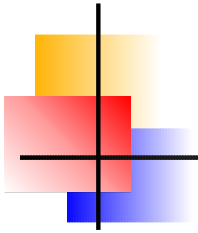
We allow so called **trivial walk**, *having only one vertex i.e. $k = 1$, i.e. walk in form (v_1).*



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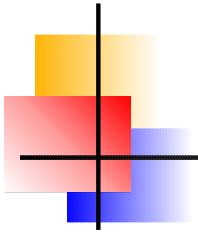

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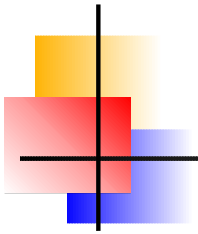

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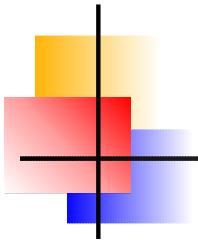

Directed walk, directed trail, directed path

Definition

Let $\overrightarrow{G} = (V, H)$ *be a digraph.*

A directed walk (a directed v_1–v_k *walk)** *in digraph* $\overrightarrow{G}$ *is arbitrary alternating sequence of vertices and arcs in the form:*

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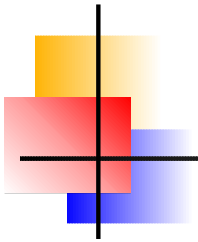

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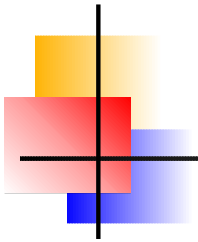

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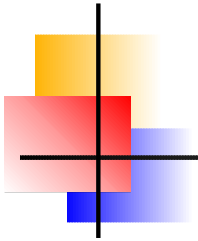

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in which every arc h_i is incident with both vertices v_i, v_{i+1} such that one of them is a head and one of the is the tail of the arc h_i.

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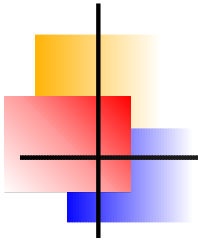

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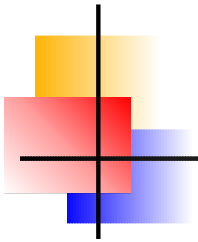

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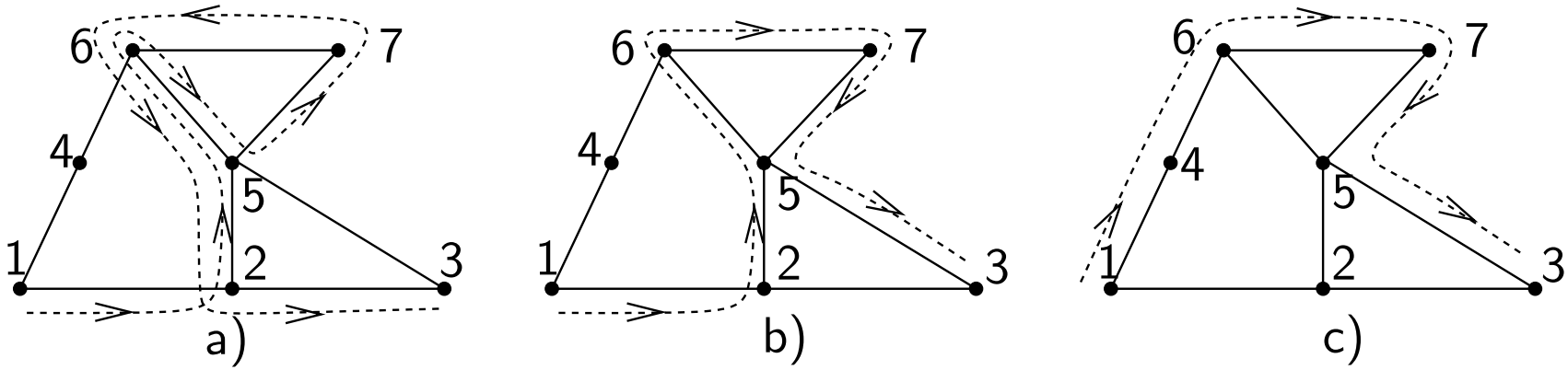

Obr.: Walk, trail an path in a graph.

a) 1–3 walk: $(1, \{1,2\}, 2, \{2,5\}, 5,$
$\{5,6\}, 6, \{6,5\}, 5, \{5,7\}, 7, \{7,6\}, 6, \{6,5\}, 5, \{5,2\}, 2, \{2,3\}, 3)$.

b) 1–3 trail: $(1, \{1,2\}, 2, \{2,5\}, 5, \{5,6\}, 6, \{6,7\}, 7, \{7,5\}, 5, \{5,3\}, 3)$.

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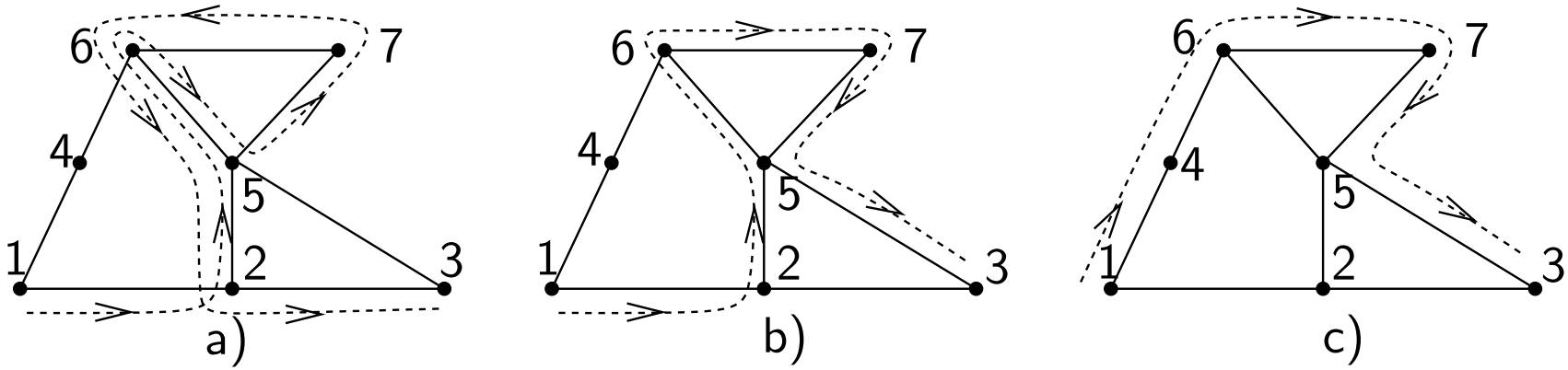

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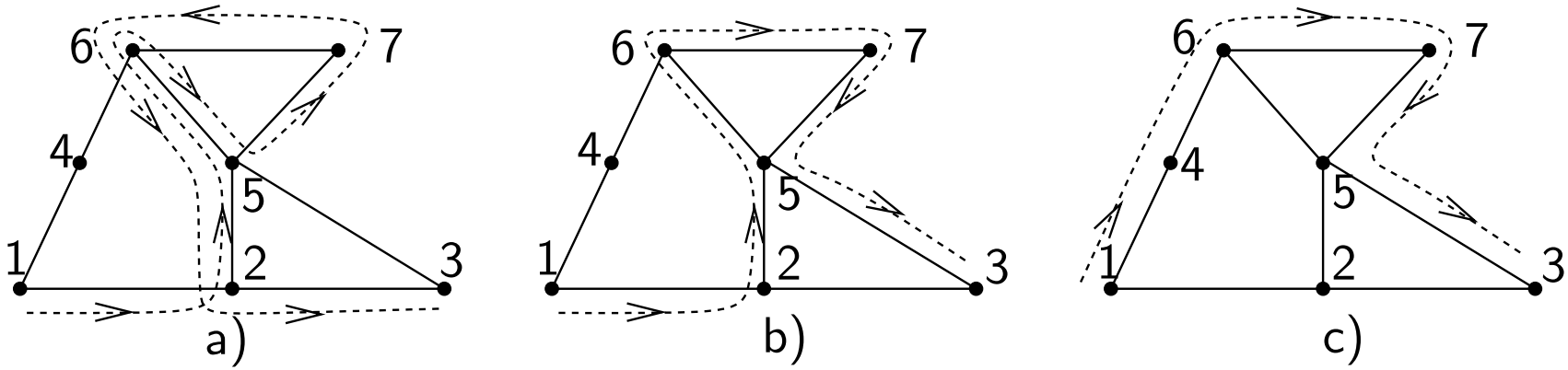

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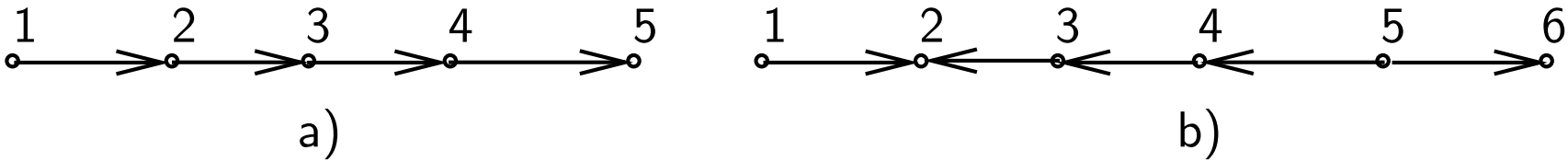

Obr.: Directed path and quasi-path in a digraph.

a) 1–5 directed paths: $(1, (1,2), 2, (2,3), 3, (3,4), 4, (4,5), 5)$.
b) 1–6 quasi-path: $(1, (1,2), 2, (3,2), 3, (4,3), 4, (4,5), 5, (5,6), 6)$.

We allow to write instead of

$$\mu(v_1, v_k) = (v_1, h_1, v_2, h_2, \ldots, v_{k-1}, h_{k-1}, v_k),$$

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namely in cases when it can not come to misunderstanding.

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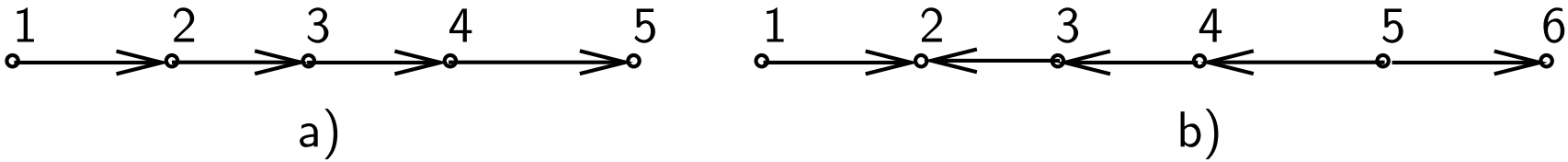

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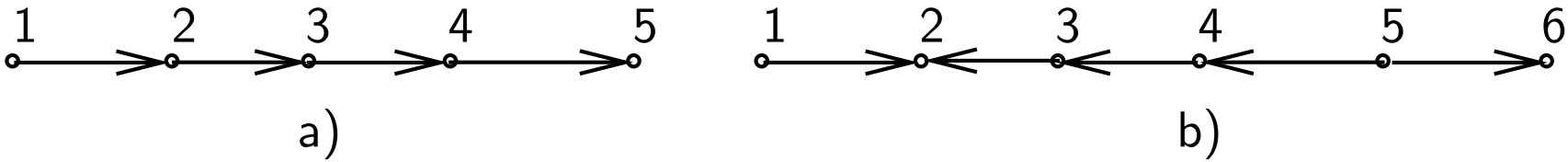

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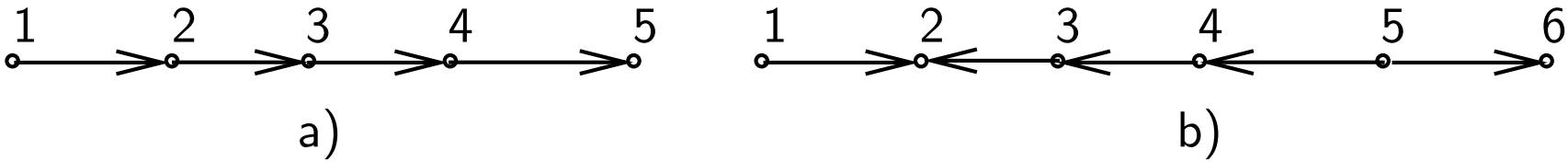

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Closed walk

Definition

A walk (quasi-walk, trail, quasi-walk)

$$\mu(v_1, v_k) = (v_1, h_1, v_2, h_2, \ldots, v_{k-1}, h_{k-1}, v_k)$$

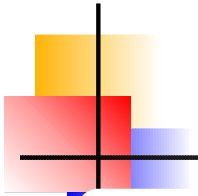
is **closed**, *if* $v_1 = v_k$.
Otherwise, a walk (quasi-walk, trail, quasi-walk) $\mu(v_1, v_k)$ *is* **open**.

Remark

Closed path can not be defined similarly since a path must not contain repeated vertices.
We have a cycle instead of closed path.

Definition

Cycle (directed cycle, quasi-cycle) *is non trivial closed walk (directed walk, quasi-walk) in which every vertex but the first and the last appears at most once.*



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Concatenation of walks

Definition

Let

$$\mu(v_1, v_r) = (v_1, \{v_1, v_2\}, v_2, \{v_2, v_3\}, v_3, \ldots, \{v_{r-1}, v_r\}, v_r),$$

$$\mu(w_1, w_s) = (w_1, \{w_1, w_2\}, w_2, \{w_2, w_3\}, w_3, \ldots, \{w_{s-1}, w_s\}, w_s),$$

let $v_r = w_1$.

Concatenation of walks $\mu(v_1, v_r)$, $\mu(w_1, w_s)$ *is the walk*

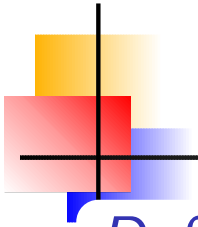
$$\mu(v_1, v_r) \oplus \mu(w_1, w_s) =$$

$$= (v_1, \{v_1, v_2\}, v_2, \ldots, \{v_{r-1}, v_r\}, v_r = w_1, \{w_1, w_2\}, w_2, \ldots, \{w_{s-1}, w_s\}, w_s).$$

Concatenation of directed walks is defined by the same way.

Remark

Concatenation $\mu(u, w) \oplus \mu(w, v)$ *of two paths* $\mu(u, w)$ $\mu(w, v)$ *need not be a paths – the result can be a walk.*



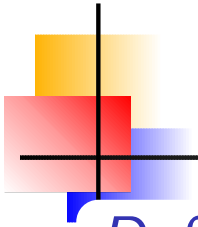
Reachability

Definition

Let $G = (V, H)$ is a graph, resp. digraph, let $u,\ v \in V$.
We say that the vertex v is je **reachable** *from the vertex u in a graph, resp. digraph G, if there exist a u–v walk in graph G, resp. if there exists a directed u–v walk in digraph G.*

Theorem

If there exists a u–v walk in graph $G = (V, H)$ then there exists a u–v path in G.



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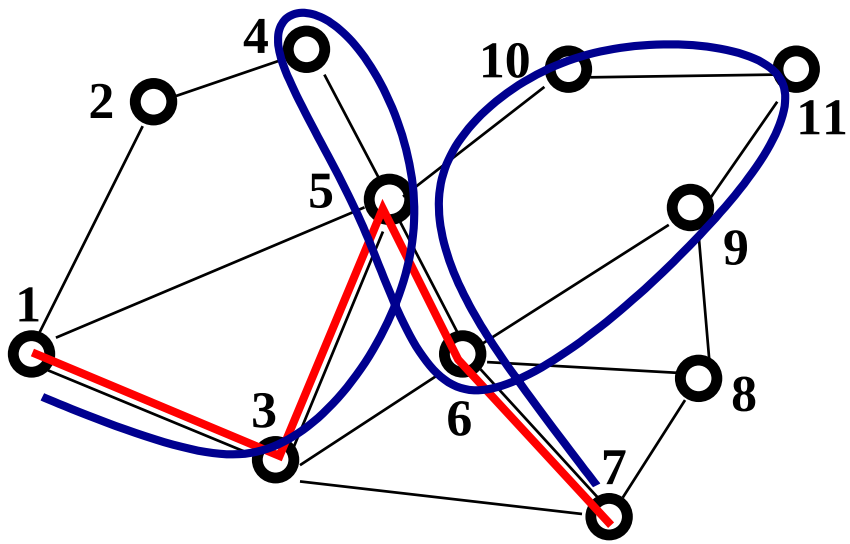

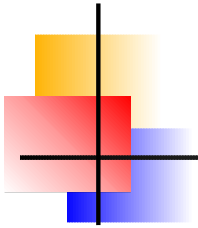

Connectivity

Definition

We say that graph $G = (V, H)$ is **connected**, *if there exists a u–v path for every pair of vertices $u, v \in V$. Otherwise, the graph G is said to be* **disonnected**.

Definition

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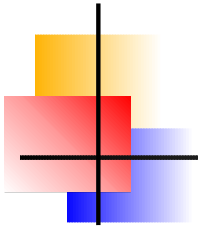

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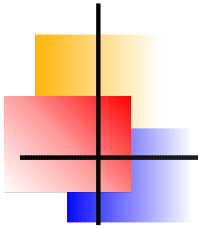

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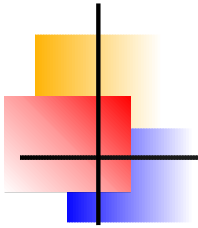

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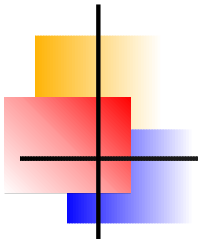


Connectivity of digraphs, component of a digraph

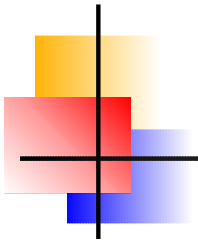
Definition

Let $\overrightarrow{G} = (V, H)$ be a digraph.
We will say that digraph $\overrightarrow{G}$ is **weakly connected**, *if there exists a u–v quasi-path in $\overrightarrow{G}$ for every pair of vertices $u, v \in V$.*
Otherwise we will say that digraph is $\overrightarrow{G}$ **disconnected**.

We will say that digraph $\overrightarrow{G}$ is **unilaterally connected** *if there exists a directed u–v path or a directed v–u path for every pair of vertices $u, v \in V$.*

Digraph $\overrightarrow{G}$ is **strongly connected**, *if there exist both a directed u–v path and a directed v–u path for every pair of vertices $u, v \in V$.*

A component of digraph $\overrightarrow{G}$ *is a maximal weakly connected subgraph of $\overrightarrow{G}$.*



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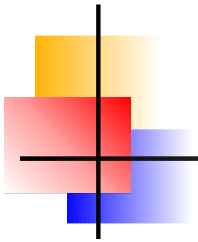

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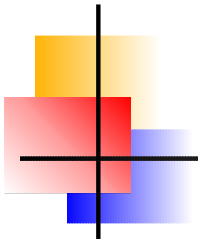

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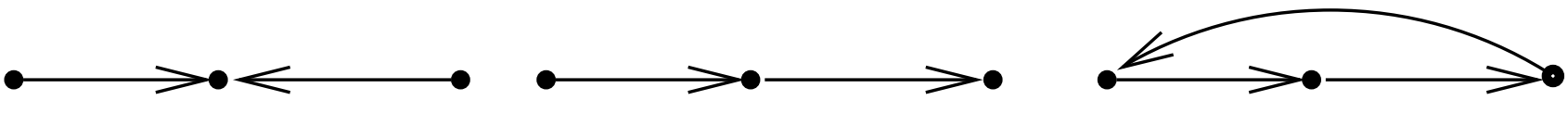
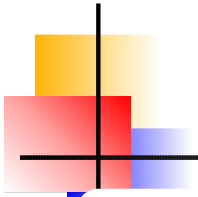
Example



Obr.: Digraphs with warious types of connesctivity.

a) weakly connected b) unilaterally connected c) strongly connected



Tarry's algorithm

Algorithm

Tarry's algorithm *for creating a closed walk in a graph* $G = (V, H)$ *starting in arbitrary vertex* $s \in V$ *and containing all edges of a component of* G *determined by vertex* s.
Resulting walk will be called **Tarry's walk**.

- **Step 1.** *Start with arbitrary vertex* $s \in V$, *set* $u := s$, $T = (u)$.
  $\{T$ *is inicialised as a trivial walk containing only one vertex.*$\}$
- **Step 2.** *If possible, choose an edge* $\{u, v\}$ *incident with last vertex* u *of the walk* T *observing undermentione rules* **T1**, **T2** *and include it into walk* T.

  Mark the direction in which the edge $\{u, v\}$ *is used in* T. *If the vertex* v *was not used in* T *till now denote the edge* $\{u, v\}$ *as the* **first access edge - FAE**.

  Rules for selection next edge:

  T1: *Every edge* $\{u, v\} \in H$ *can be used in one direction only once (i.e. once in direction* $\{u, v\}$ *and once in dirction* $\{v, u\}$*).*
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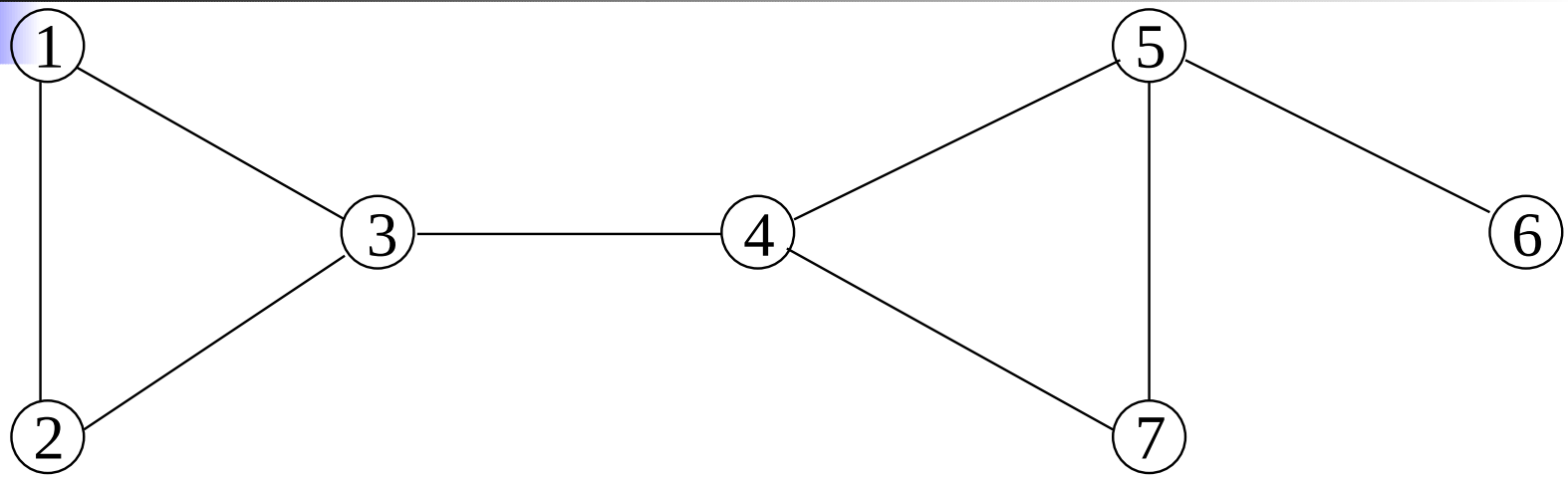

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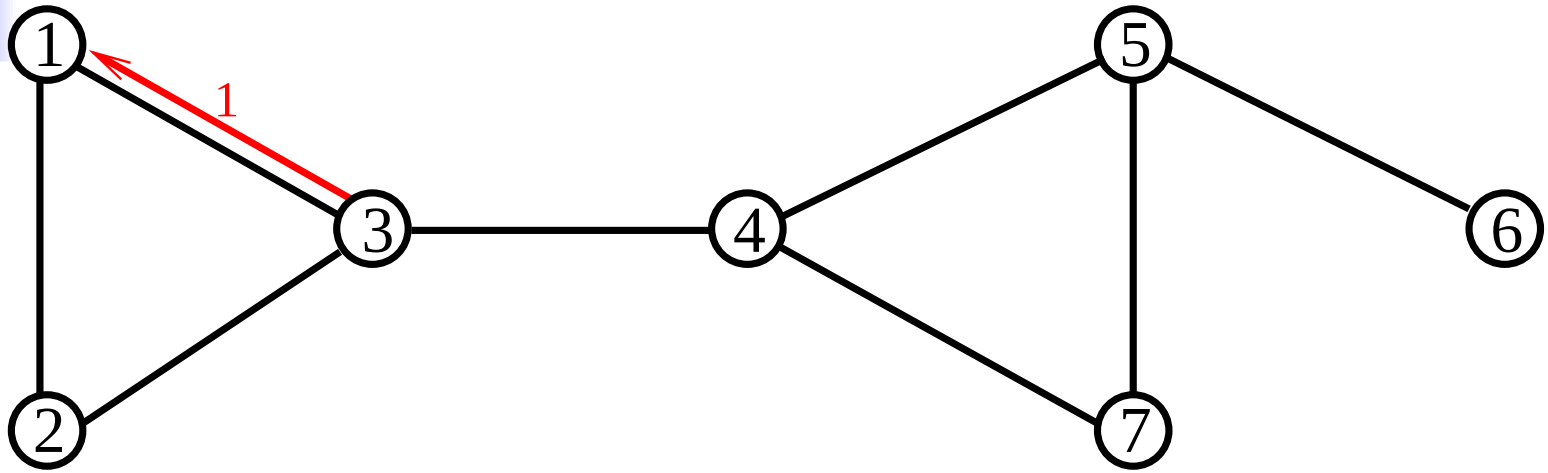

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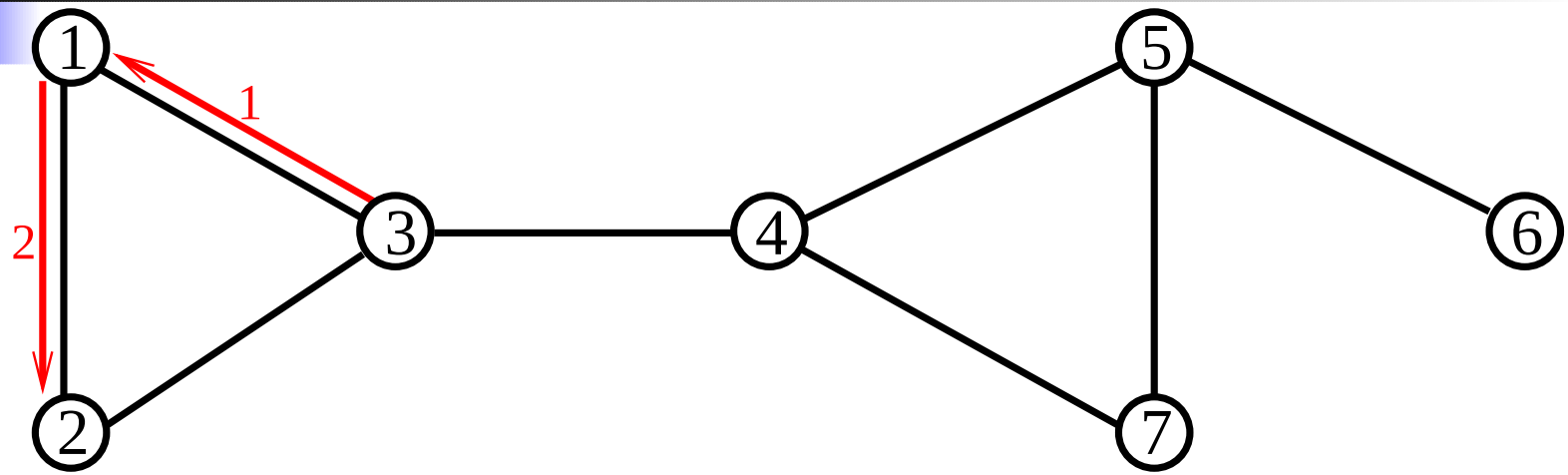

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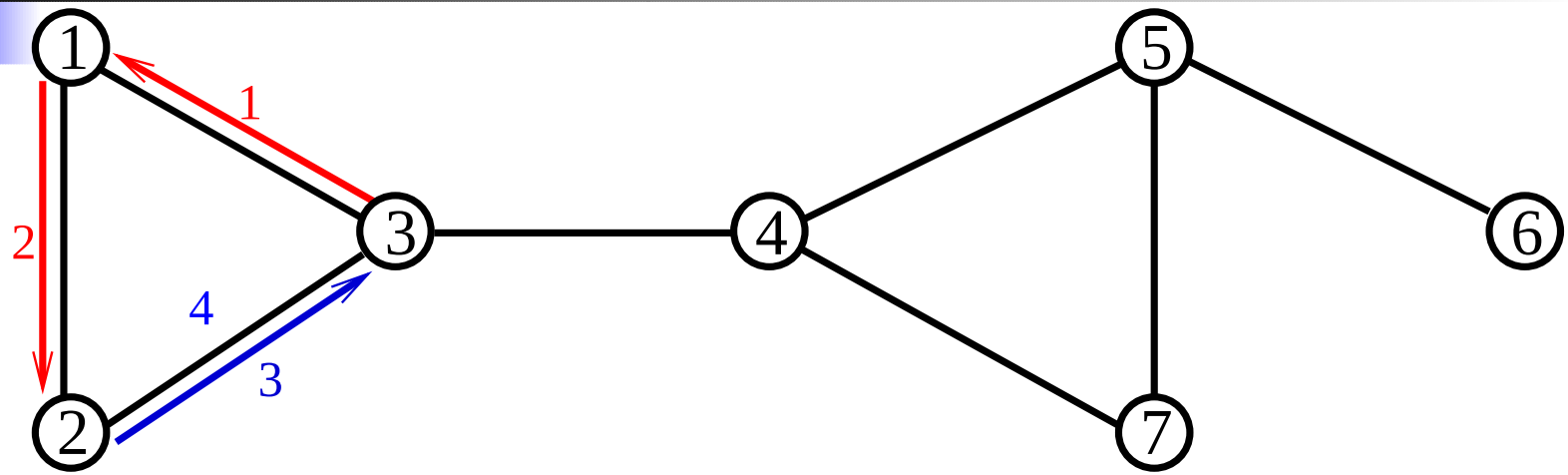

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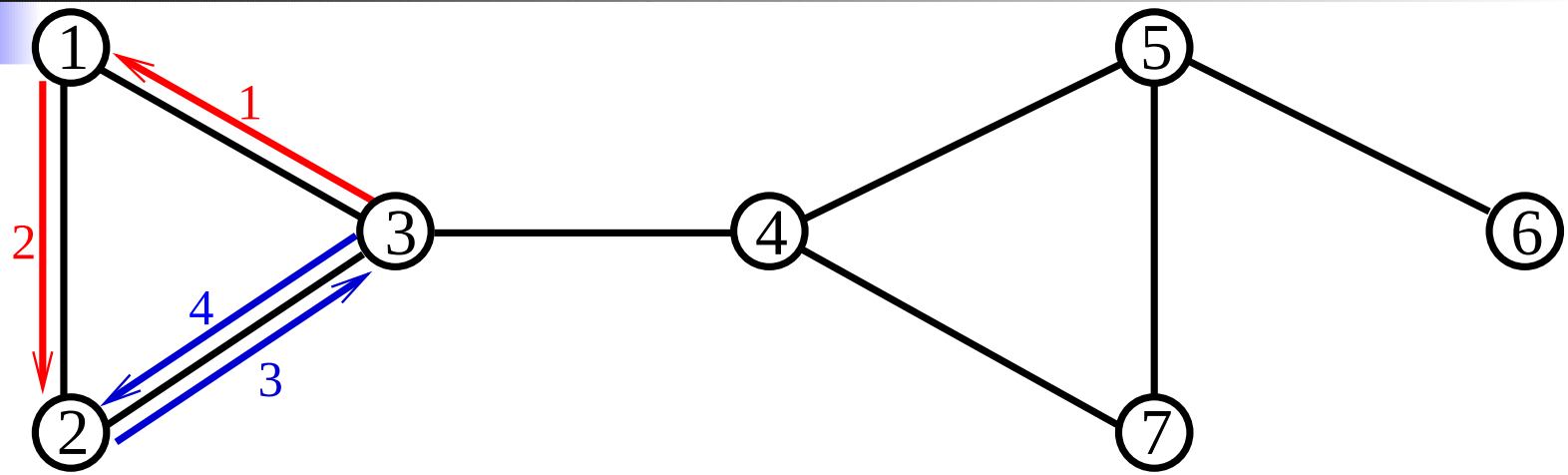

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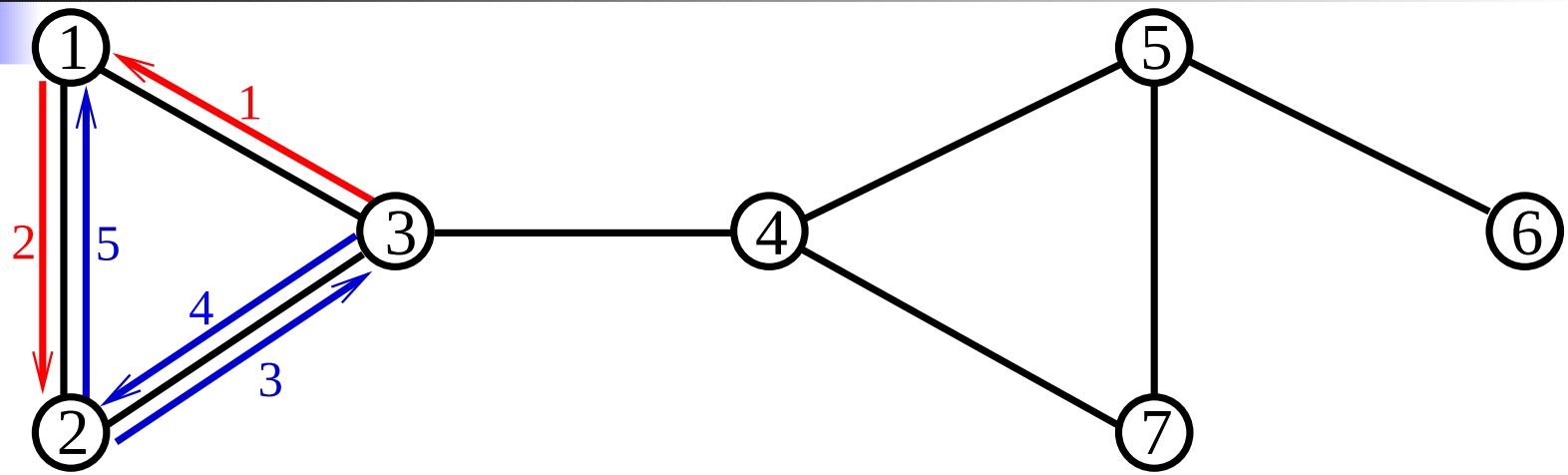

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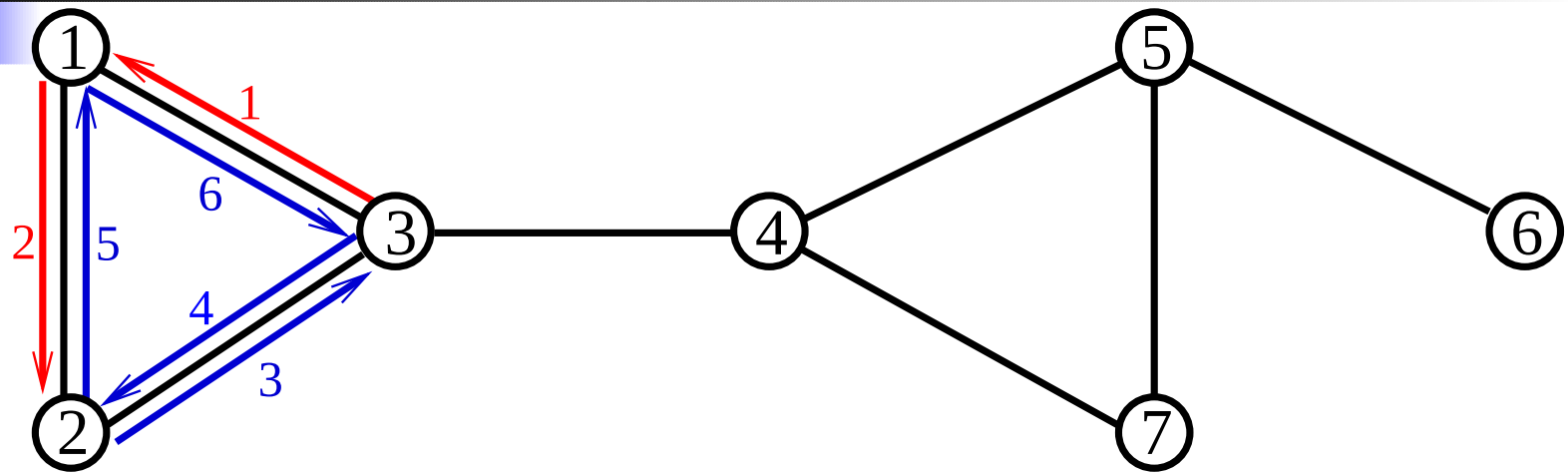

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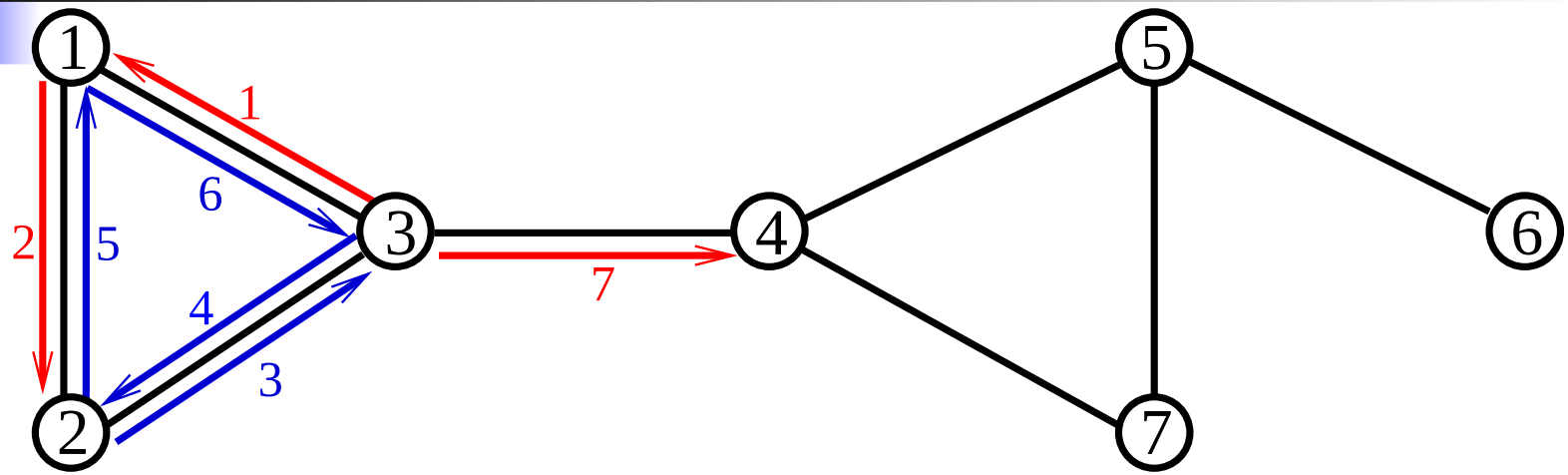

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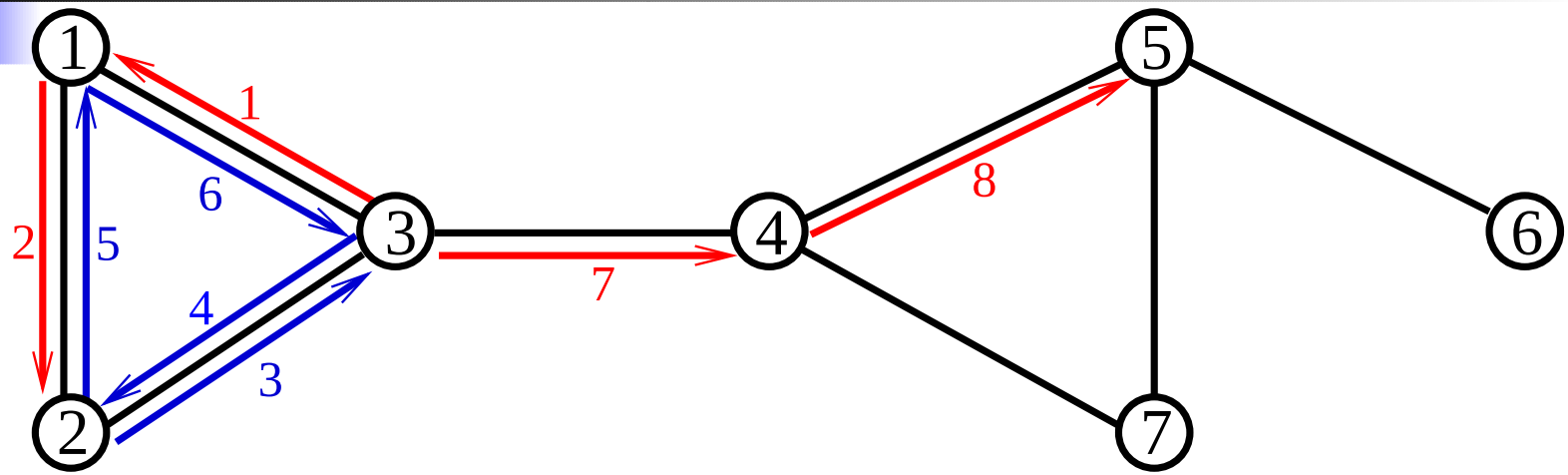

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9	{5,6}							⇒							•	
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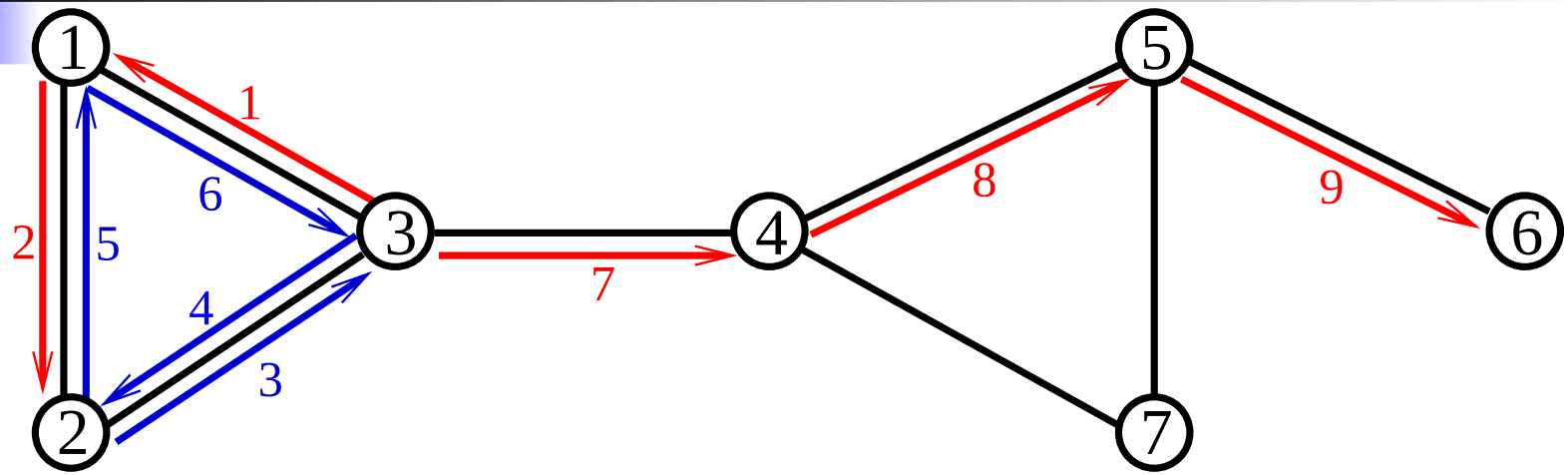
Example



r		{1,2}	{1,3}	{2,3}	{3,4}	{4,5}	{4,7}	{5,6}	{5,7}	1	2	3	4	5	6	7
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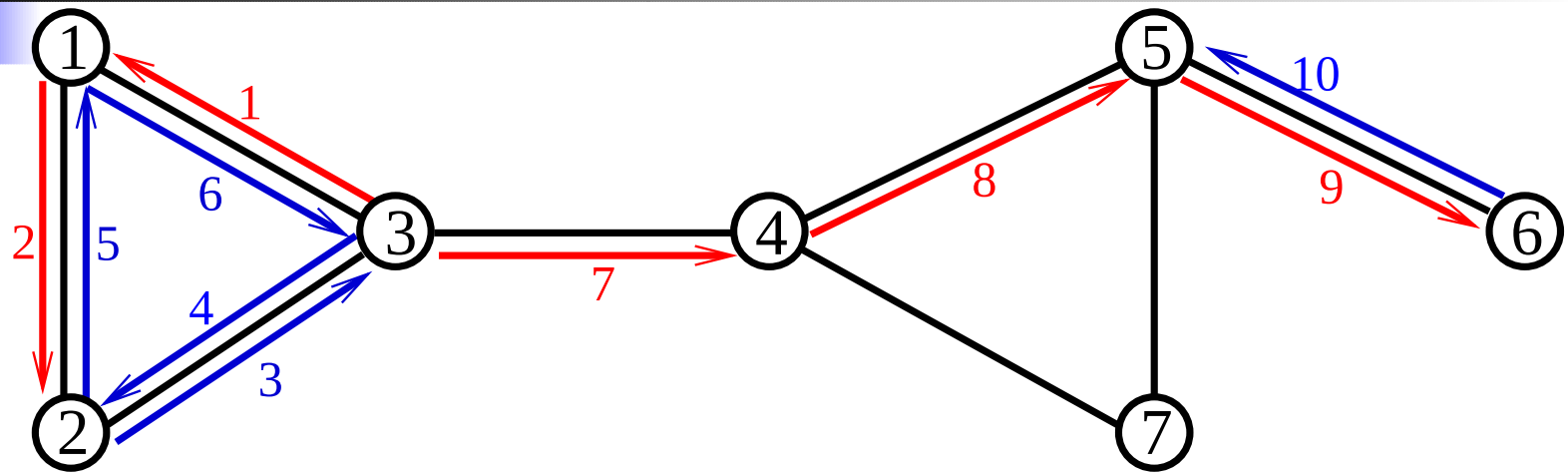

r		{1,2}	{1,3}	{2,3}	{3,4}	{4,5}	{4,7}	{5,6}	{5,7}	1	2	3	4	5	6	7
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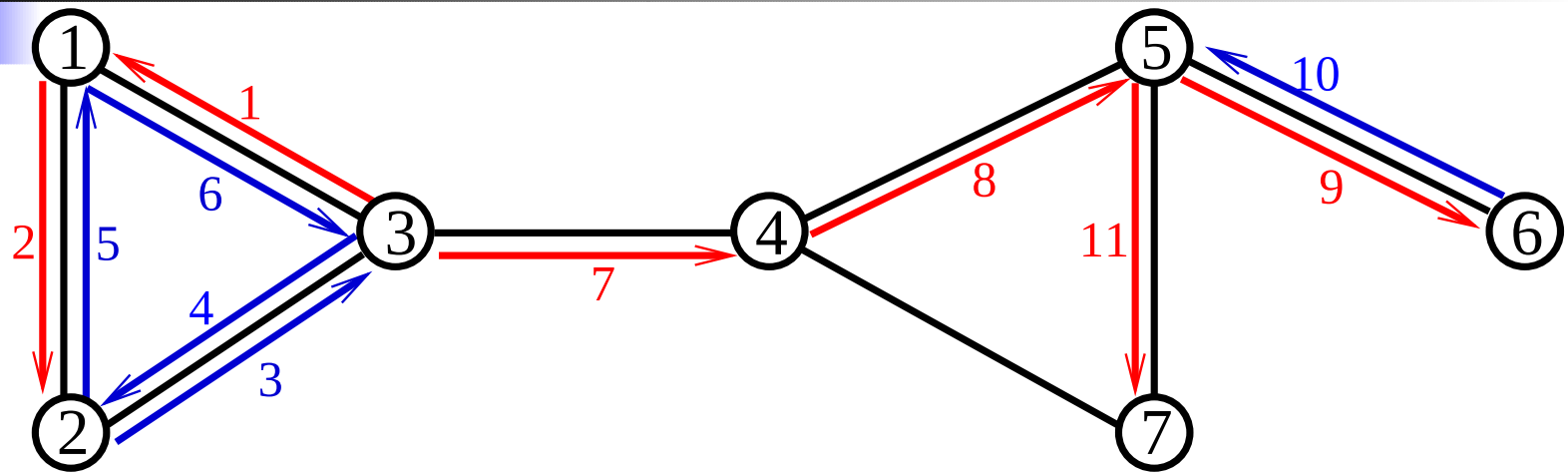

r		{1,2}	{1,3}	{2,3}	{3,4}	{4,5}	{4,7}	{5,6}	{5,7}	1	2	3	4	5	6	7
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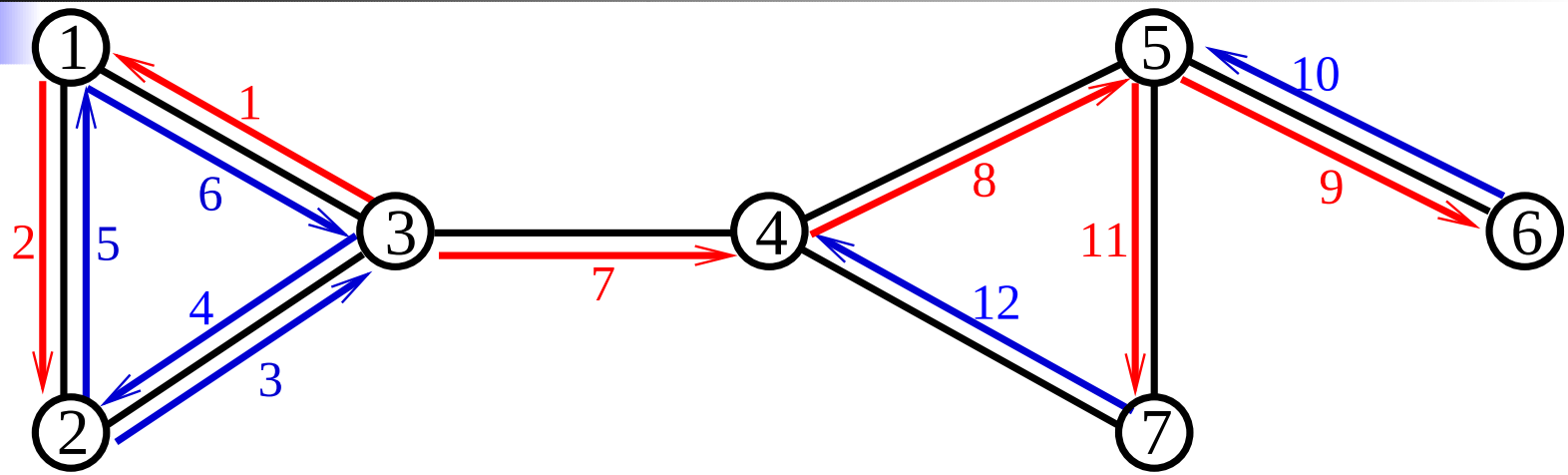

r		{1,2}	{1,3}	{2,3}	{3,4}	{4,5}	{4,7}	{5,6}	{5,7}	1	2	3	4	5	6	7
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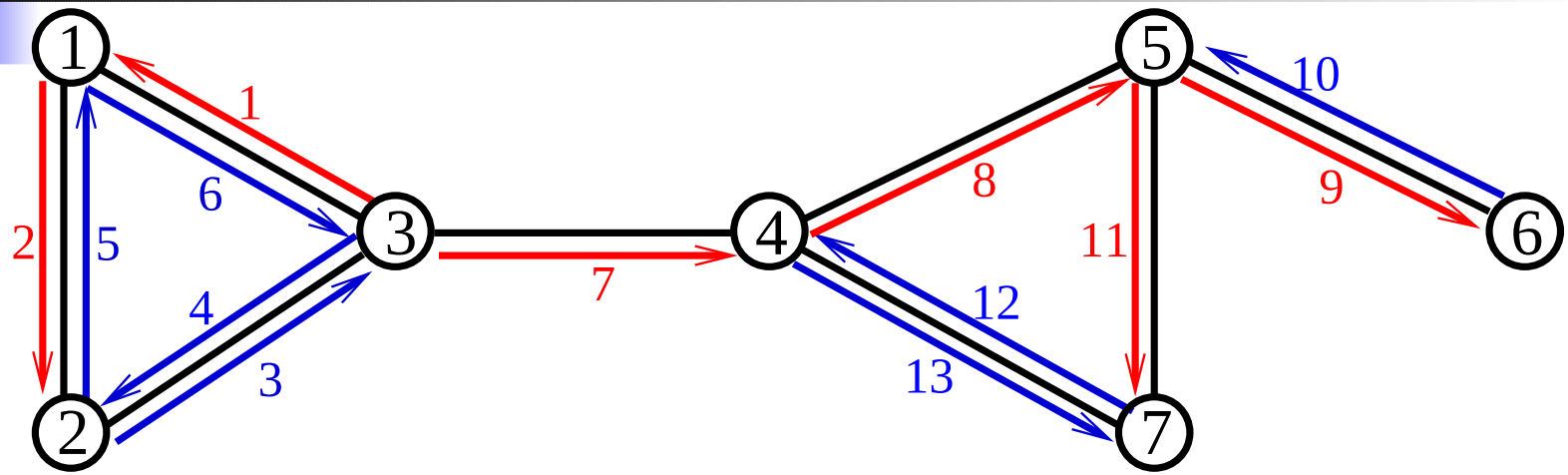

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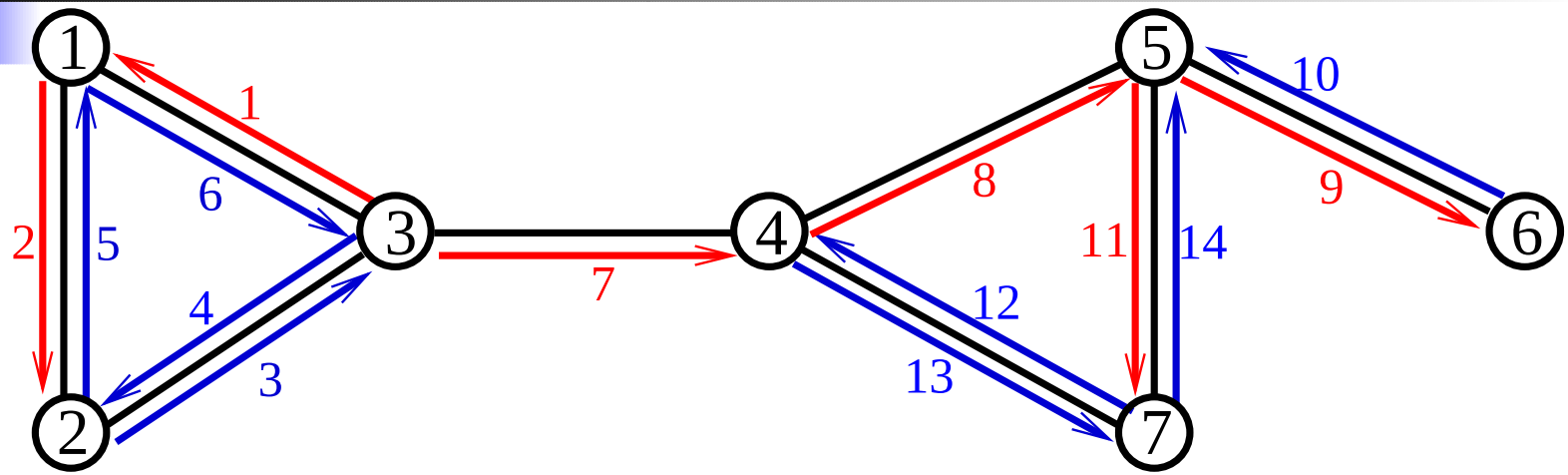

r		{1,2}	{1,3}	{2,3}	{3,4}	{4,5}	{4,7}	{5,6}	{5,7}	1	2	3	4	5	6	7
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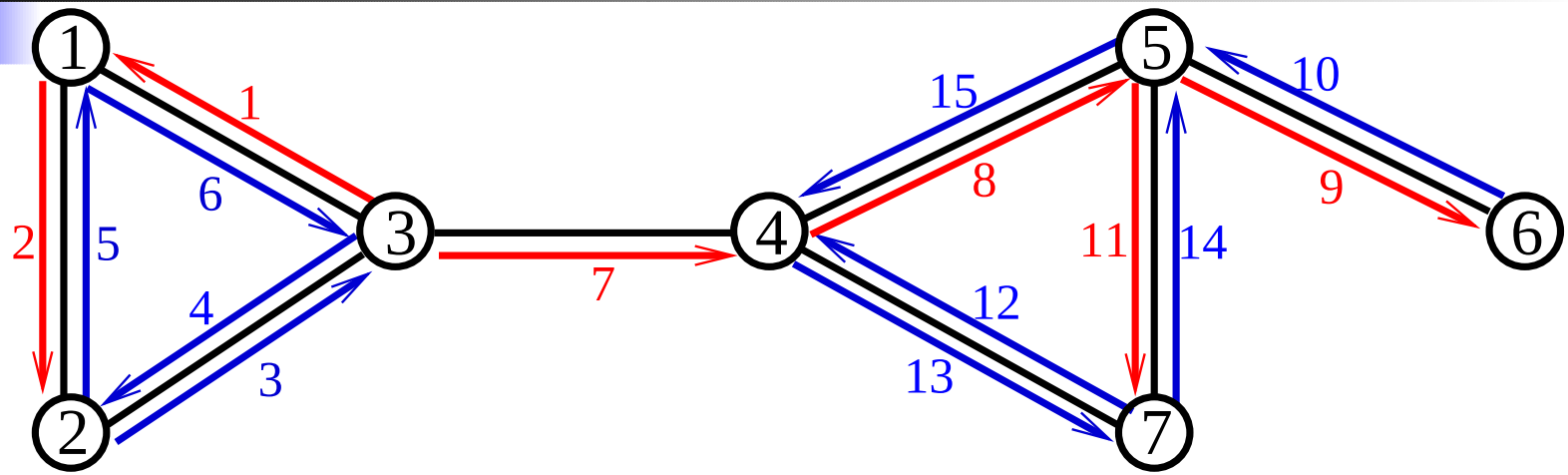
Example



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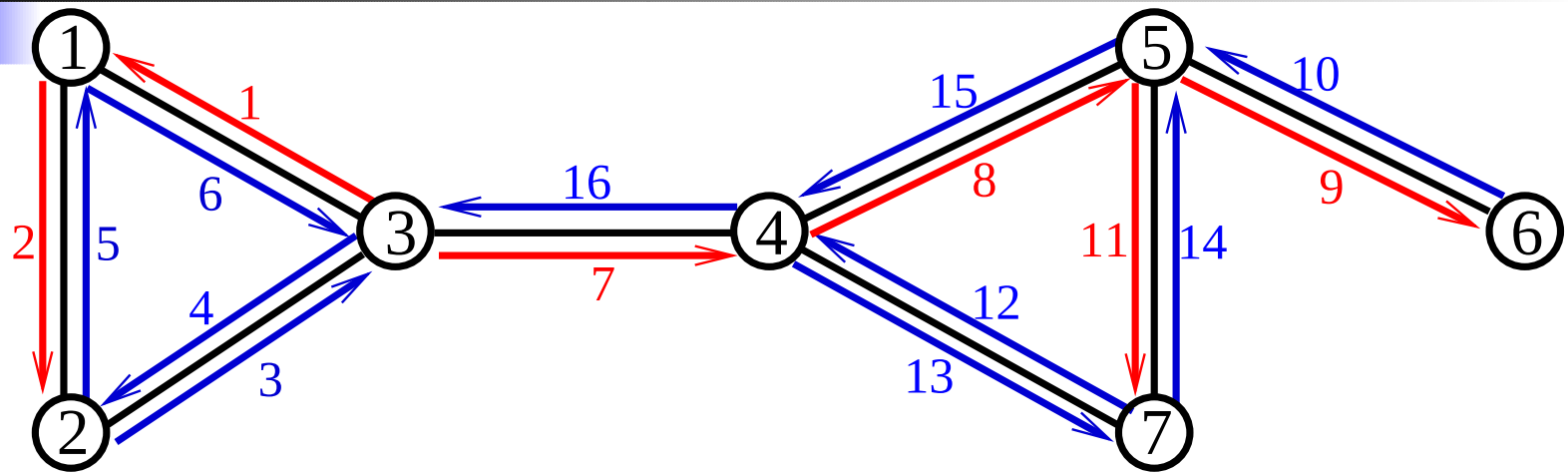

r		{1,2}	{1,3}	{2,3}	{3,4}	{4,5}	{4,7}	{5,6}	{5,7}	1	2	3	4	5	6	7
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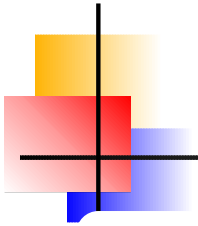


Length of a walk

Definition

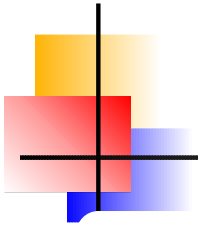
Let $\mu(u,v)$ be a u–v walk (resp. directed walk resp. quasi-walk) in an edge weighted graph $G = (V, H, c)$ (resp. digraph $\overrightarrow{G} = (V, H, c)$).
The length of the walk (resp. quasi-walk) $\mu(u,v)$ *or* **the cost of the walk (resp. quasi-walk)** *is the sum of edge weights of all edges in $\mu(u,v)$, wheras the edge weight is added so many times how many times corresponding edge occurs in the walk $\mu(u,v)$.*
The length of the walk $\mu(u,v)$ will be denoted by $d(\mu(u,v))$.

Remark

Definition of a walk allows also trivial walk having only one vertex. The length of such a walk is zero.

Remark

Last definition defines also the length of trail, path, cycle and all directed variants of these notions, since all are special cases of walk.



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Dĺžka ťahu, dĺžka cesty

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Definition of length can be simplified for trail, cycle etc. i.e. for instances of walk with no repeated edges as follows:

The length $d(\mu(u,v))$ of a trail (quasi-trail, path or quasi-path) $\mu(u,v)$ is the sum of edge weights of all edges of $\mu(u,v)$, i. e.

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Sometimes it is useful to define the length of a walk $\mu(u,v)$ in a graph $G = (V,H)$, resp. digraph $\overrightarrow{G} = (V,H)$, which is not edge weighted. In mentioned case, the length of a walk $\mu(u,v)$ is defined as the number of edges of the walk $\mu(u,v)$.
Such defined length of walk is identical with the length of the walk in an edge weighted graph (resp digraph) $G' = (V,H,c)$, where $c(h) = 1$ for every edge $h \in H$.[a]

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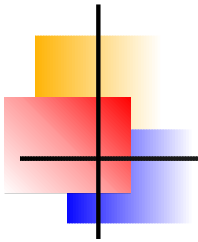


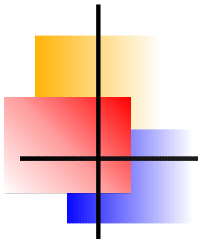
Shortest path in graph and digraph

Definition

Shortest *u-v* **path** *in an edge weighted graph* $G = (V, H, c)$ *(resp. in an arc weighted digraph* $\overrightarrow{G} = (V, H, c)$*) is that u-v path in* G *(resp. that directed u-v path in* $\overrightarrow{G}$*), which has the least length.*

Agreement

- *All shortest path algorithms will be formulated* **for arc weighted digraphs** $\overrightarrow{G} = (V, H, c)$ *with nonnegative arc weight* $c(h) \geq 0$.
- *We will suppose that* $0 \notin V$.



Shortest path in graph

Shortet path in a non directed graph $G = (V, H, c)$ can be found as the shortest path in digraph $\overrightarrow{G} = (V, \overrightarrow{H}, \overrightarrow{c})$, in which $\overrightarrow{H}$ is the set of directed edges containing for every non directed edge $h = \{u, v \in H\}$ two directed edges (u, v), (v, u), both with the same edge weight equal to $c(h) = c(\{u, v\})$, i.e.

$$\overrightarrow{c}(u, v) = \overrightarrow{c}(v, u) = c(h).$$

Fundamental shortest path algorithm

Algorithm

Fundamental point-to-all shortest path algorithm. *This algorithm will find all shortest u–v directed paths from a fixed vertex $u \in V$ into all reachable vertices $v \in V$ in an edge weighted digraph $\overrightarrow{G} = (V, H, c)$ with nonnegative edge weight $c(h)$ (and where $0 \notin V$).*

- **Step 1. Initialisation.**
  Assign two labels $t(i)$ a $x(i)$ for every vertex $i \in V$.
  {Label $t(i)$ is upper estimate of the till now the best found u–i path and $x(i)$ is it's last but one vertex.}
  Set $t(u) := 0$, $t(i) := \infty$ for $i \in V$, $i \neq u$ a $x(i) := 0$ for all $i \in V$.
- **Step 2.** *Find out if there exists an arc $(i, j) \in H$, for which it holds*

$$t(j) > t(i) + c(i, j). \tag{3}$$

  If $(i, j) \in H$ is an arc for which it holds (3), set:

$$t(j) := t(i) + c(i, j), \quad x(j) := i$$

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- **Step 3.** *If for no arc from $(i, j) \in H$ holds (3), STOP.*

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Fundamental shortest path algorithm

The shortest u–i path can be constructed backwards using labels $x(i)$ as the path going trough vertices

$$i, x(i), x(x(i)), x(x(x(i))), \ldots, u$$

This shortest path can be written in form:

$$(u, \ldots, x(x(x(i))), \underbrace{(x(x(x(i))), x(x(i)))}_{\text{arc}}, x(x(i)),$$

$$\underbrace{(x(x(i)), x(i))}_{\text{arc}}, x(i), \underbrace{(x(i), i)}_{\text{arc}}, i)$$

Finite value of label $t(i)$ after algorithm stopped represents the length of the shortest u–i path.

If $t(i) = \infty$, the the vertex i is unreachable from vertex u.

Notation of shortest path

Agreement (about notation)

Array $x(\ )$ contains after finishing a shortest path algorithm for every vertex $i \in V$, such that $t(i) < \infty$ pointer to the last but one vertex of the shortest u-i path.
Suppose that $x(\ )$ is the array of pointers obtained by a shortest path algorithm. We define recursively $x^{(k)}(j)$ for $j \in V$ as follows:

- $x^{(1)}(j) = x(j)$
- $x^{(k)}(j) = x(x^{(k-1)}(j))$

$$x^{(k)}(j) = \underbrace{x(x(\ldots x(}_{k\text{-times}} j) \ldots))$$

Then the sequence of vertices of the shortest u-j path can be written in reverse order like this:

$$j,\ x^{(1)}(j),\ x^{(2)}(j),\ \ldots\ x^{(k)}(j) \equiv u,$$

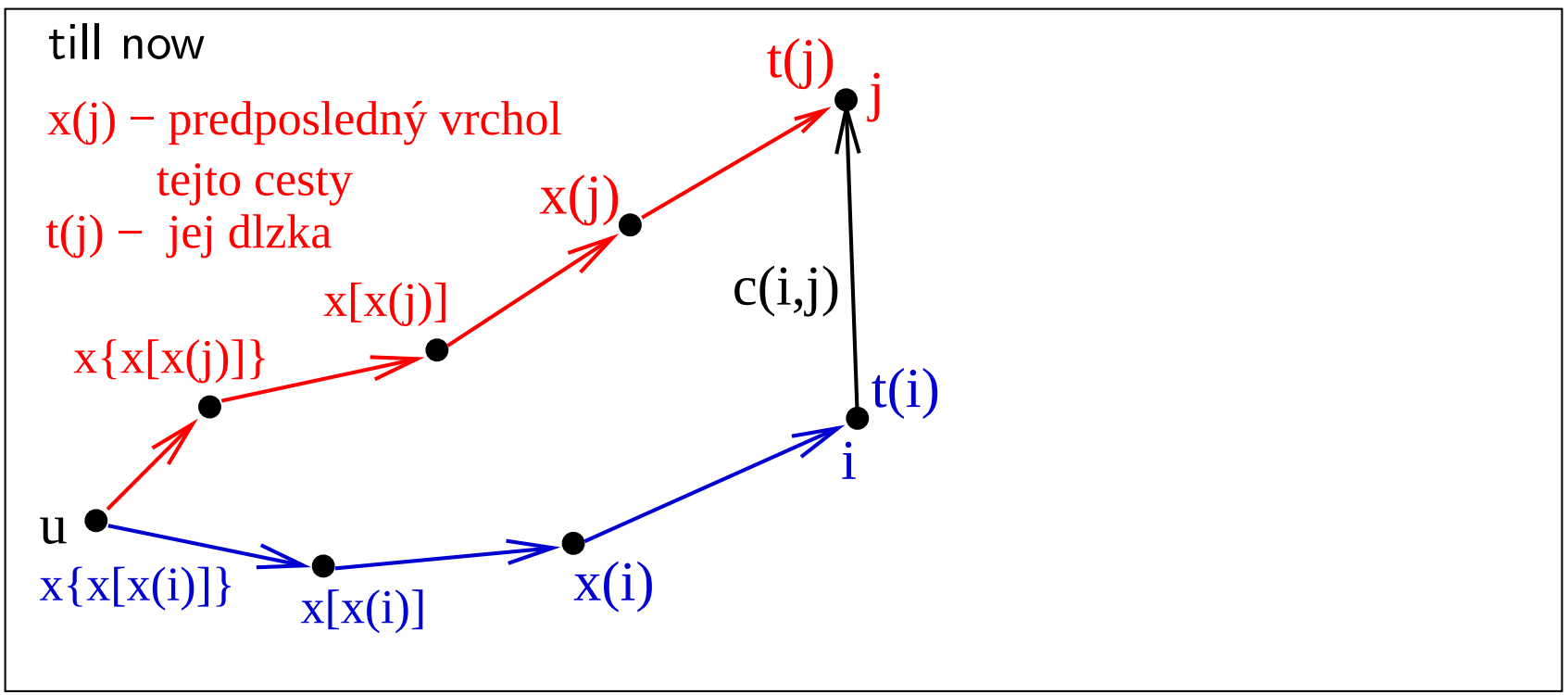
hence the shortest u-j path traverses through vertices

$$u \equiv x^{(k)}(j),\ x^{(k-1)}(j),\ \ldots,\ x^{(2)}(j),\ x^{(1)}(j),\ j$$

Principle of fundamental algorithm

Theorem

Fundamental algorithm will stop after a finite number of steps for arbitrary digraph $\overrightarrow{G} = (V, H, c)$ *with nonegative arc weight.*



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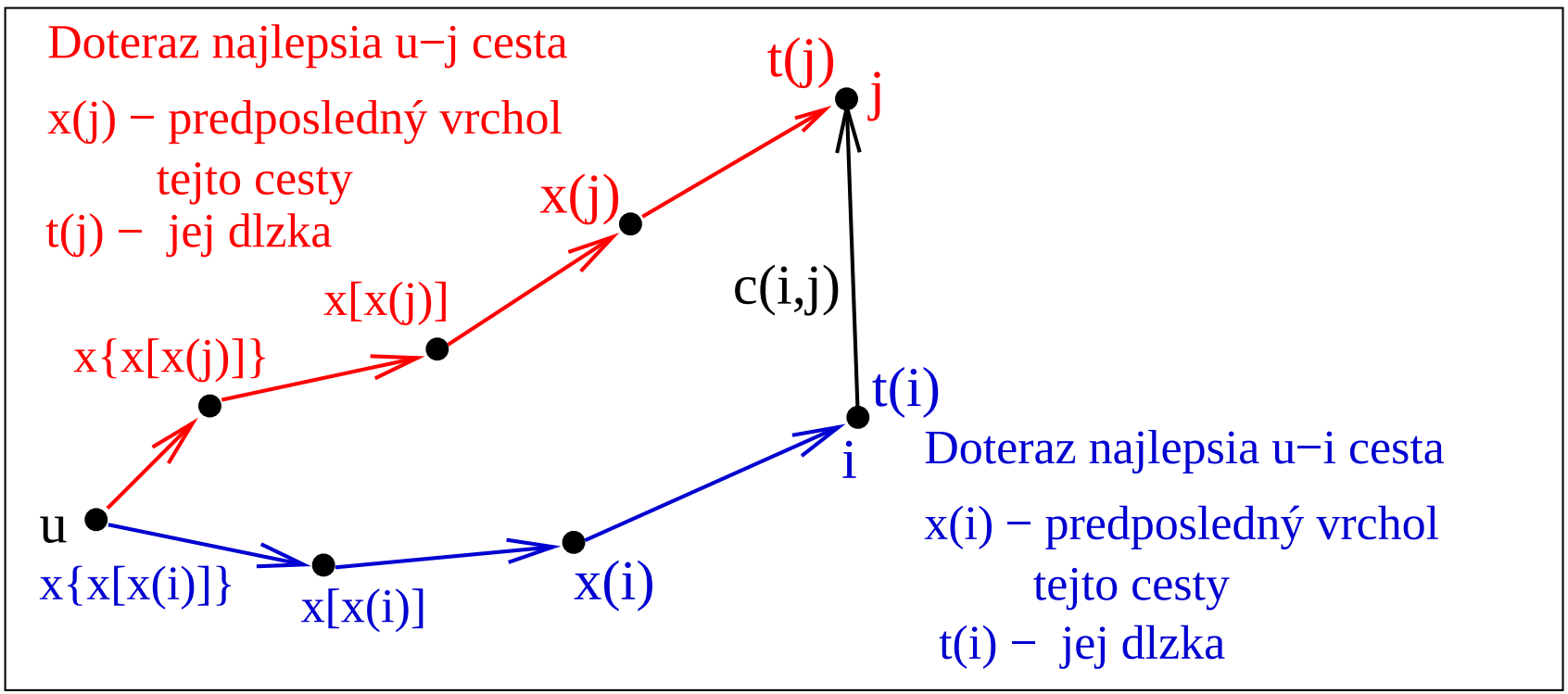

Principle of fundamental algorithm

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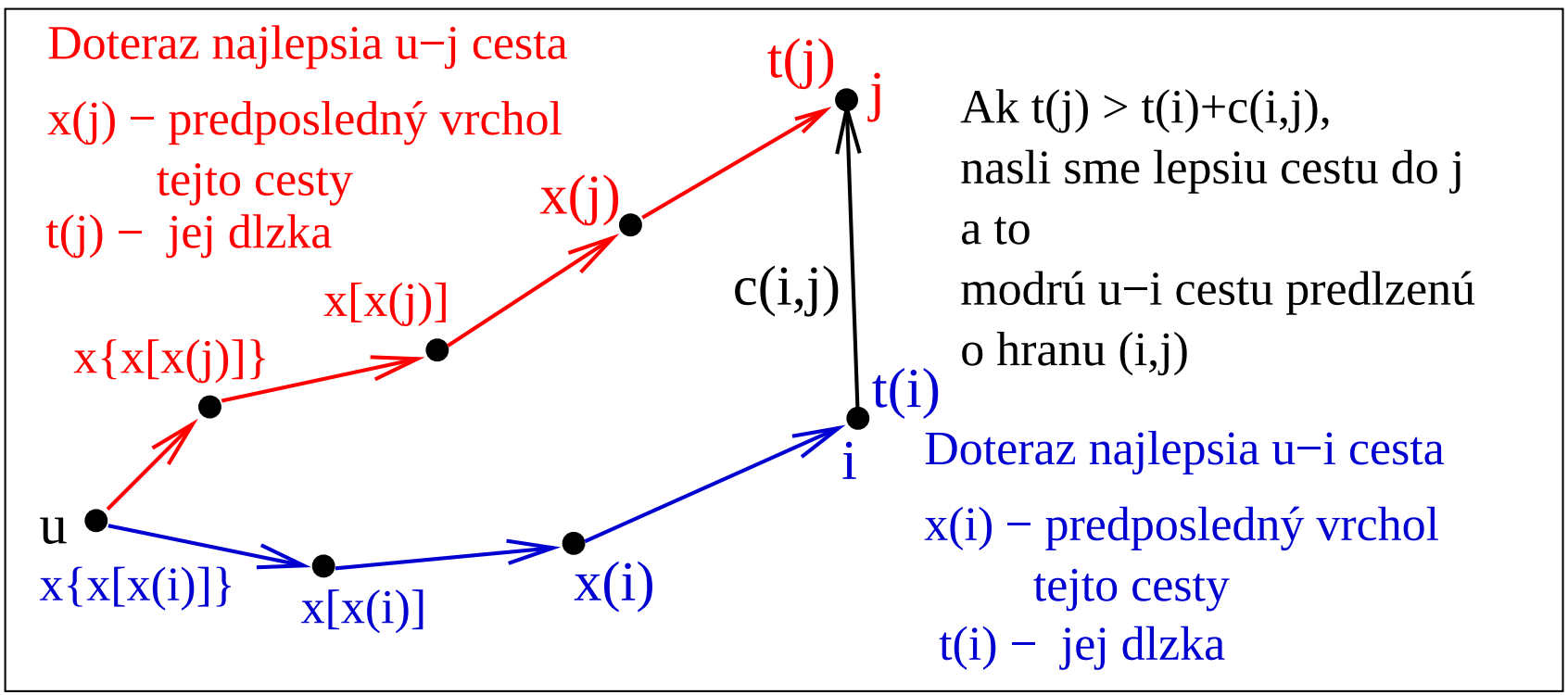

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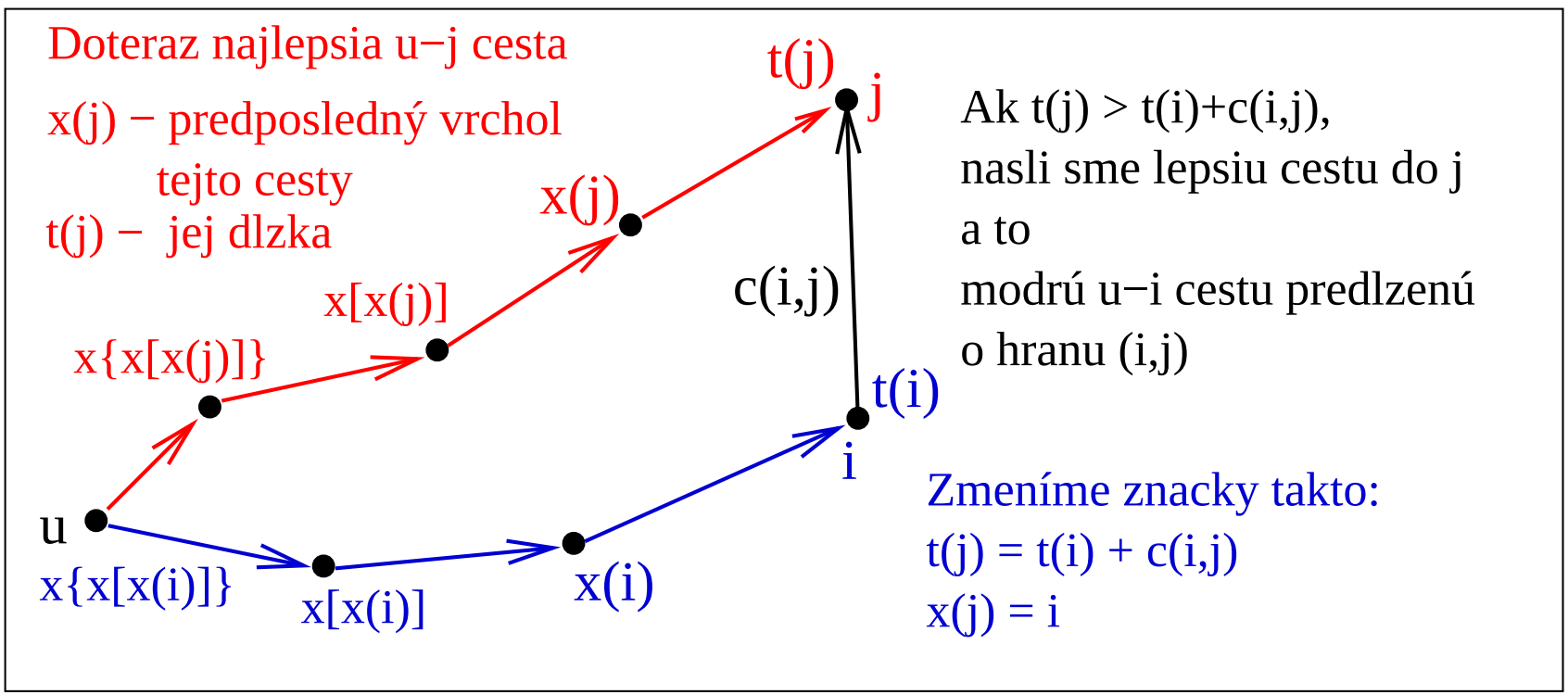

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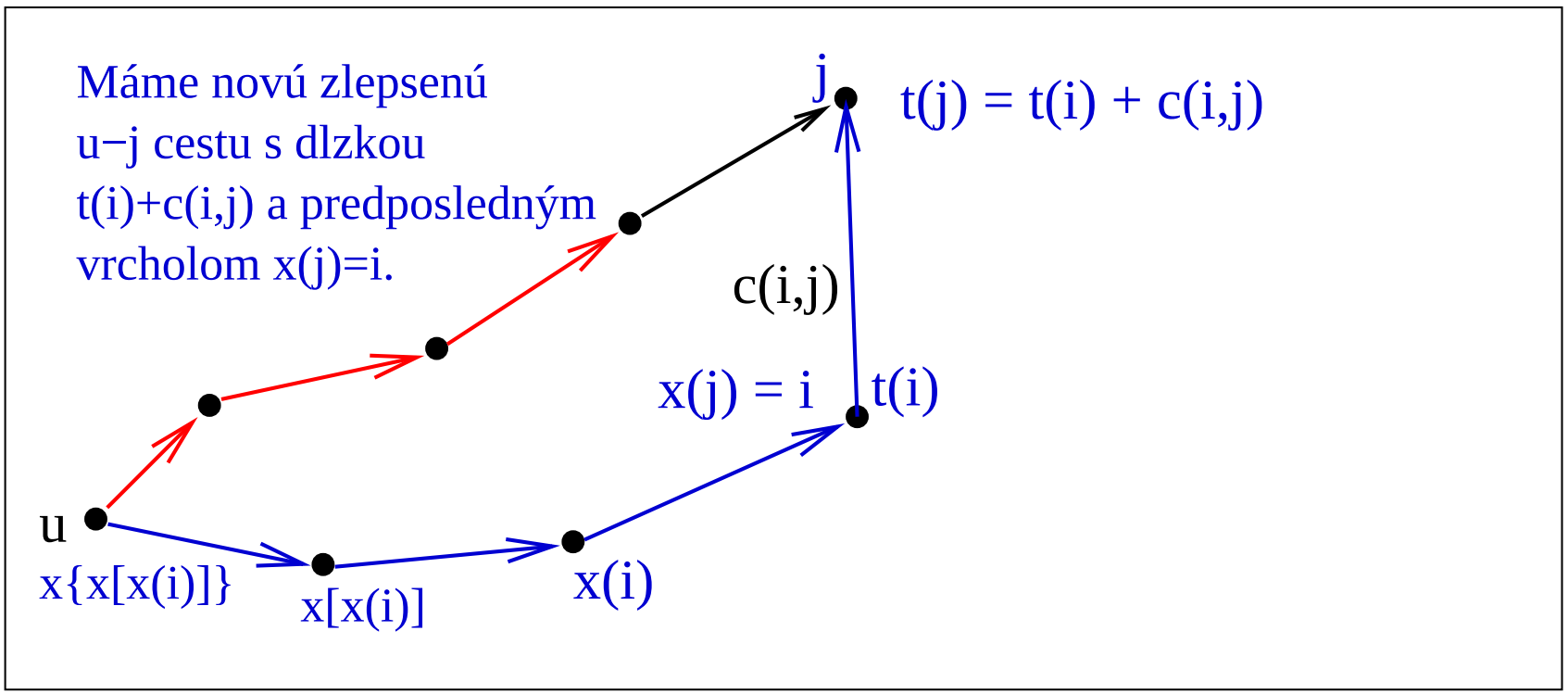
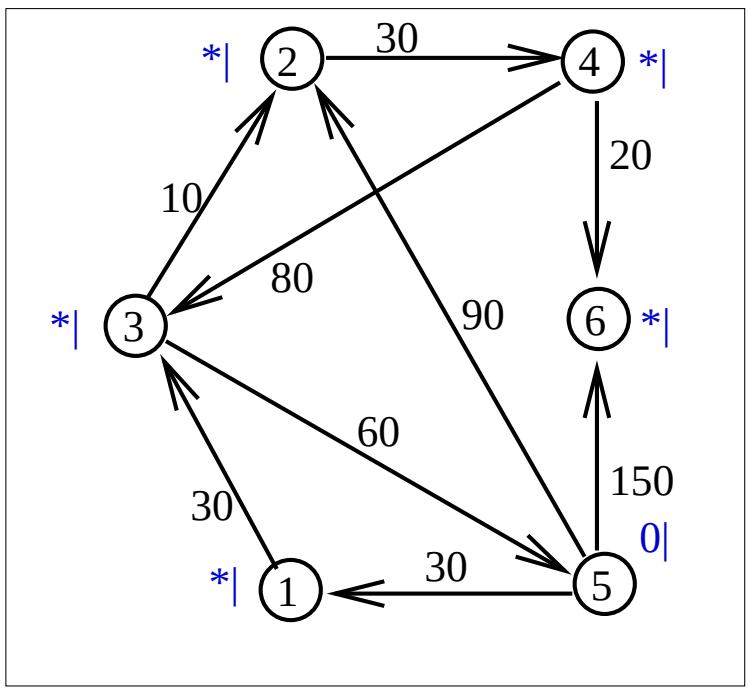


Example. Searching for all shortest paths for vertex 5.

Arc weight table for digraph $\overrightarrow{G}$

h	$(1,3)$	$(2,4)$	$(3,2)$	$(3,5)$	$(4,3)$	$(4,6)$	$(5,1)$	$(5,2)$	$(5,6)$
$c(h)$	30	30	10	60	80	20	30	90	150



$h=(i,j)$	$t(i)$	c(h)	1	2	3	4	5	6
			$t(v)\|x(v)$					
-			$\infty\|$	$\infty\|$	$\infty\|$	$\infty\|$	$0\|$	$\infty\|$
$(5,1)$	0	30	$30\|5$					
$(5,2)$	0	90		$90\|5$				
$(5,6)$	0	150						$150\|5$
$(1,3)$	30	30			$60\|1$			
$(2,4)$	90	30				$120\|2$		
$(3,2)$	60	10		$70\|3$				
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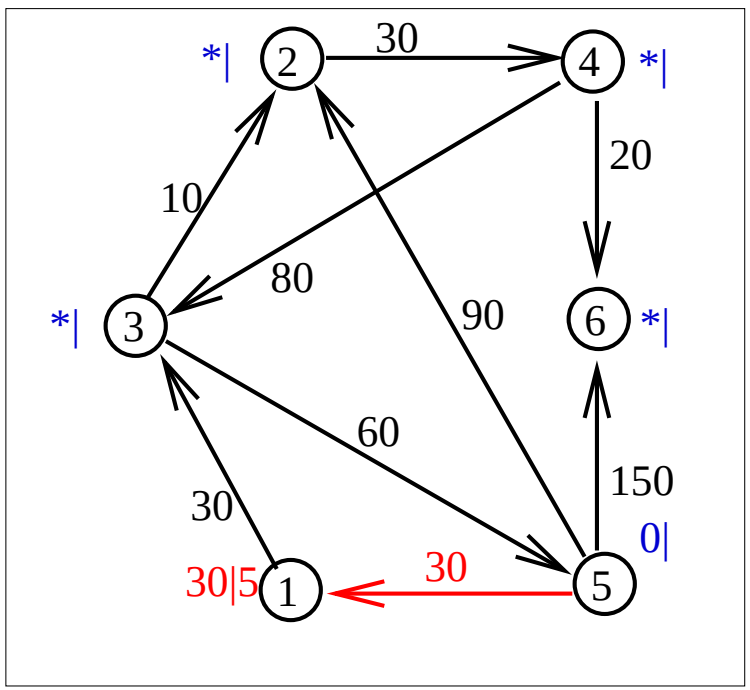

$h=(i,j)$	$t(i)$	c(h)	1	2	3	4	5	6
			$t(v)\|x(v)$					
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$(5,2)$	0	90		$90\|5$				
$(5,6)$	0	150						$150\|5$
$(1,3)$	30	30			$60\|1$			
$(2,4)$	90	30				$120\|2$		
$(3,2)$	60	10		$70\|3$				
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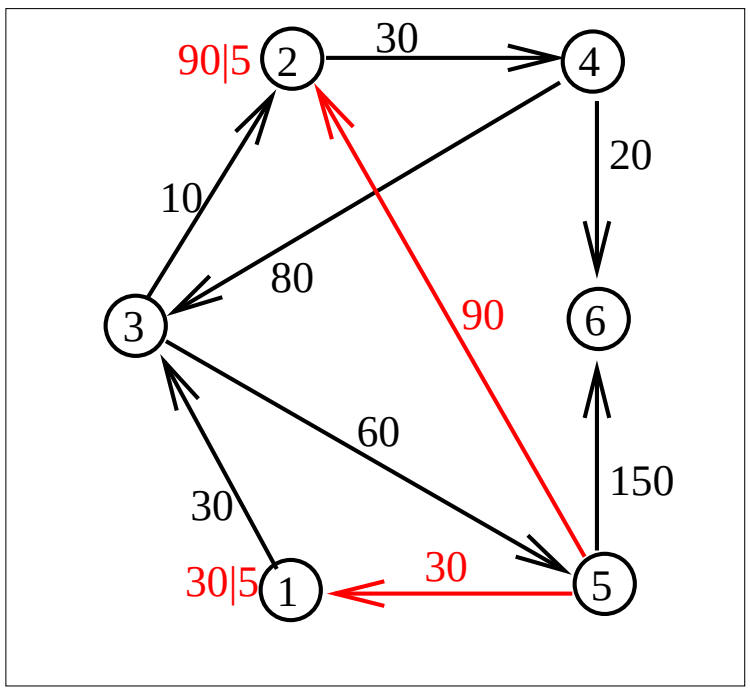

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			$t(v)\|x(v)$					
-			$\infty\|$	$\infty\|$	$\infty\|$	$\infty\|$	$0\|$	$\infty\|$
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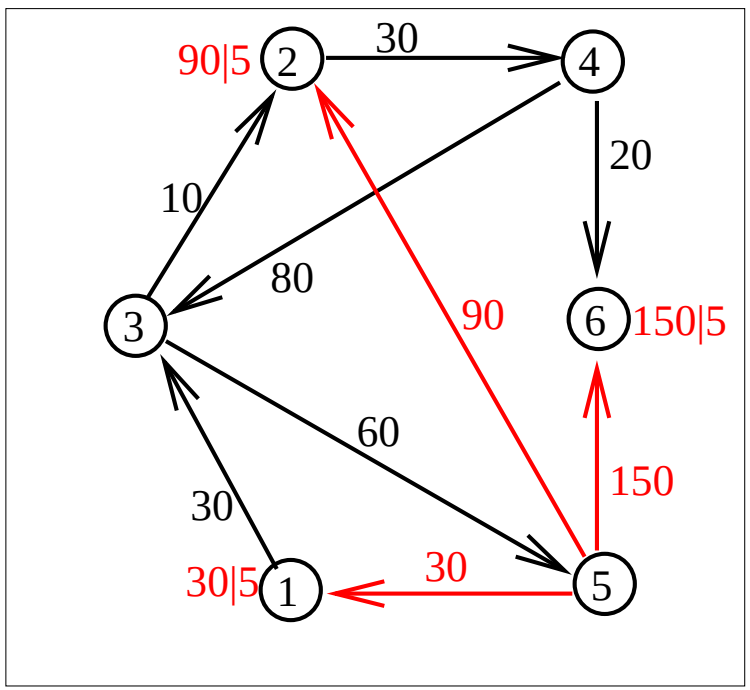

$h=(i,j)$	$t(i)$	c(h)	1	2	3	4	5	6
			$t(v)\|x(v)$					
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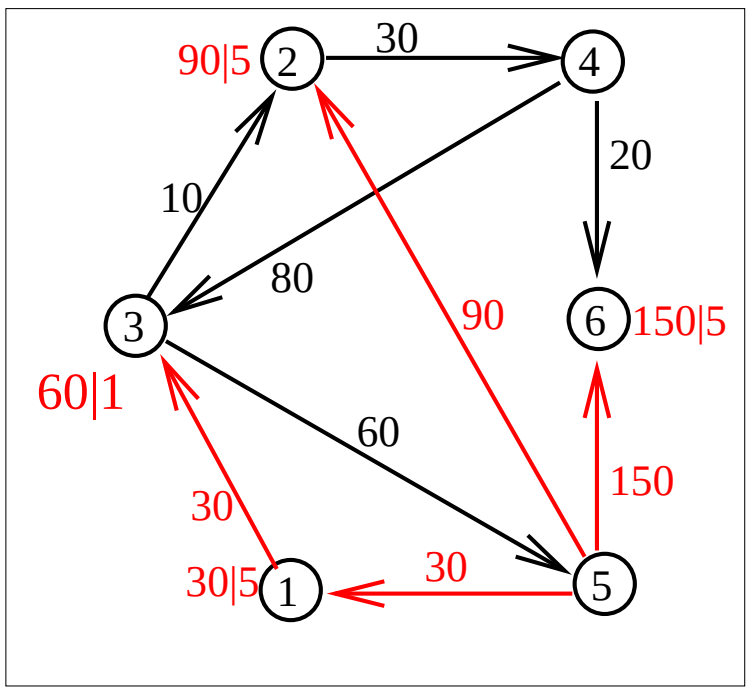

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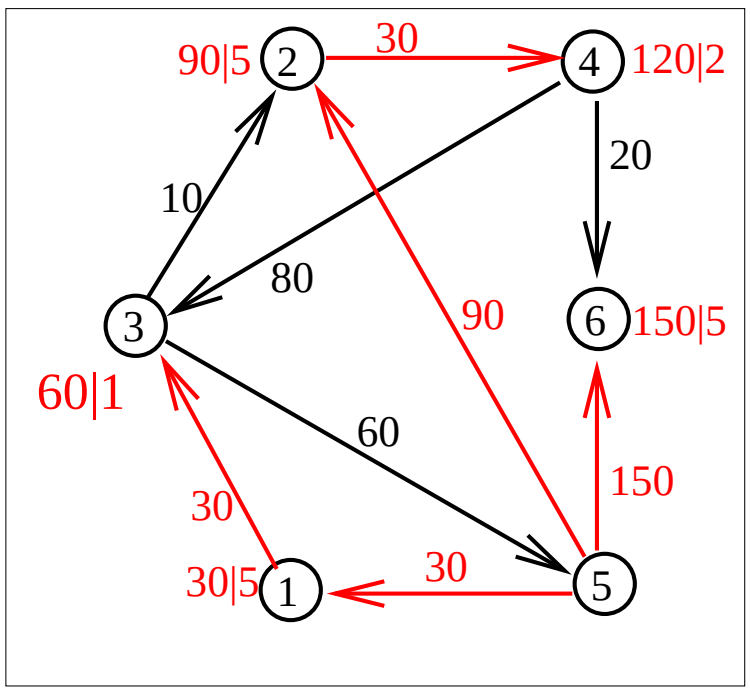

$h=(i,j)$	$t(i)$	c(h)	1	2	3	4	5	6
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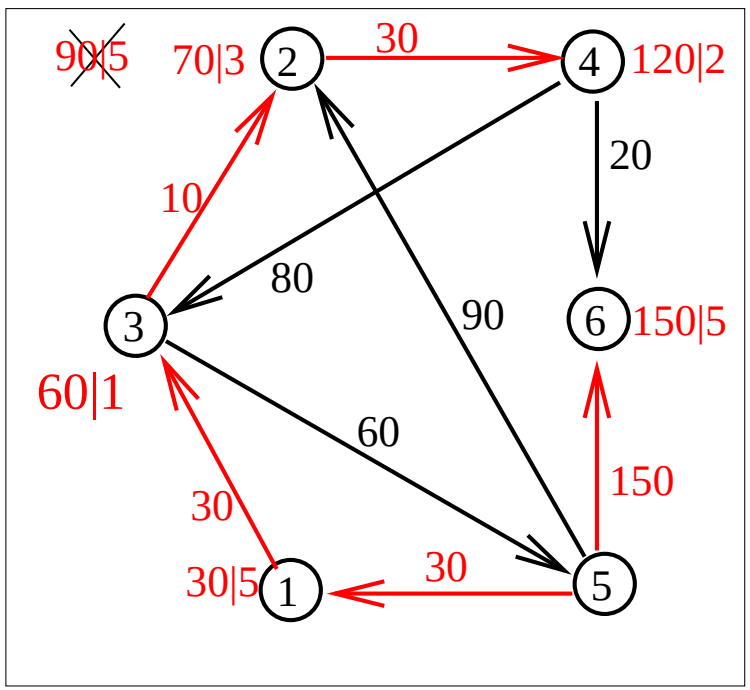

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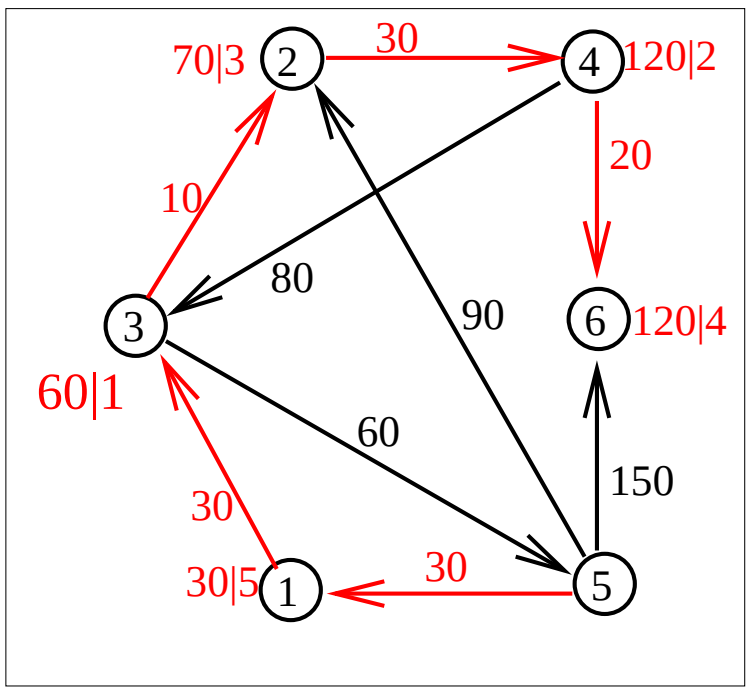

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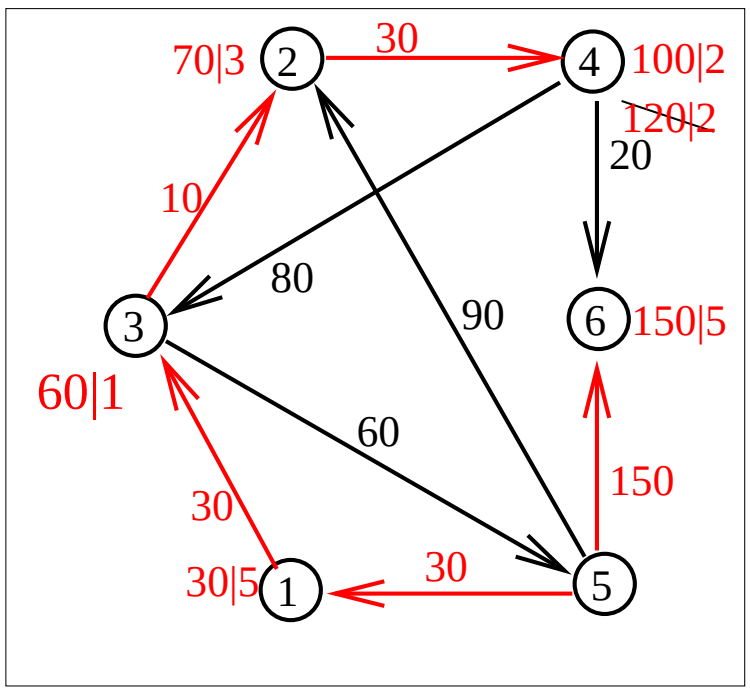

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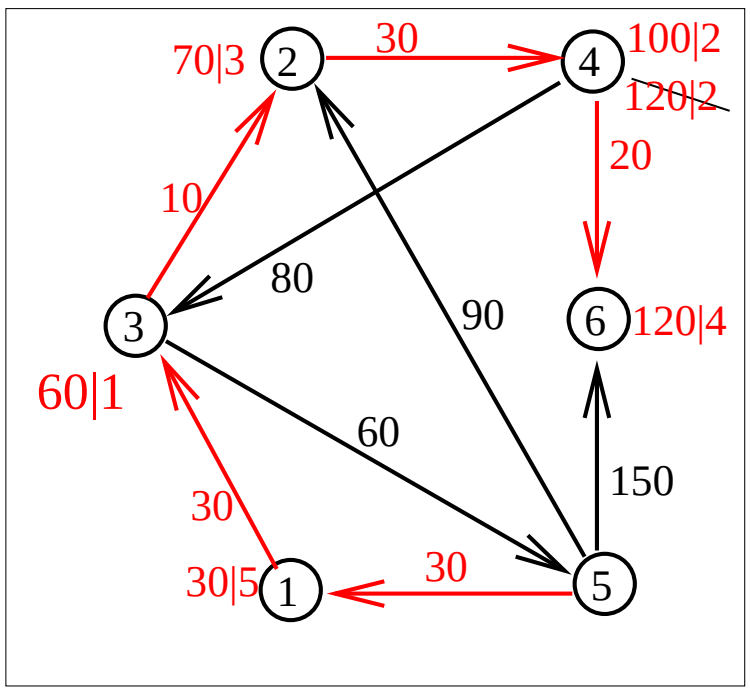

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-			$\infty\|$	$\infty\|$	$\infty\|$	$\infty\|$	$0\|$	$\infty\|$
$(5,1)$	0	30	30|5					
$(5,2)$	0	90		90|5				
$(5,6)$	0	150						150|5
$(1,3)$	30	30			60|1			
$(2,4)$	90	30				120|2		
$(3,2)$	60	10		70|3				
$(4,6)$	120	20						140|4
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Algorithm

Dijkstra's point-to-point shortest path algorithm

This algorithm will find shortest u–v directed path from a fixed vertex $u \in V$ into a fixed vertex $v \in V$ in an arc weighted digraph $\overrightarrow{G} = (V, H, c)$ with nonnegative arc weight $c(h)$ (and where $0 \notin V$).

- **Step 1.** *Initialization.*
  Assign two labels $t(i)$ a $x(i)$ for every vertex $i \in V$.
  {Label $t(i)$ is upper estimate of the till now the best found u–i path and $x(i)$ is it's last but one vertex.}

  Every label $t(i)$ can be temporary – it can change in the course of computing, or it can be final – it contains the length of shortest u–i path and it can not change.

  Set $t(u) := 0$, $t(i) := \infty$ for $i \in V$, $i \neq u$ and $x(i) := 0$ for all $i \in V$.

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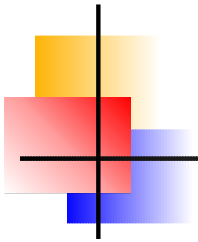


Dijkstra's algorithm

Algorithm (- continued)

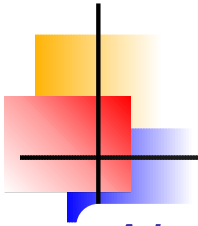
- **Step 2.** *If $r = v$, STOP.*

  If $t(v) < \infty$, then the label $t(v)$ is equal to the length of the shortest u–v path, which can be constructed in reverse order from vertex z v using ponters $x(i)$.

  Otherwise for all arcs $(r, j))$ such that $(r, j) \in H^+(r)$ do:

  If $t(j) > t(r) + c(r, j)$, then $t(j) := t(r) + c(r, j)$, $x(j) := r$.

  Keep all chainged labels as temporary.



Dijkstra's algorithm

Algorithm (- continued)

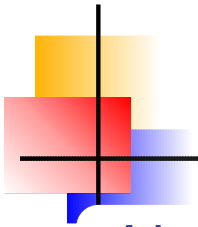
- **Step 3.** *Find vertex i with minimal temporary label $t(i)$.*

  Declare the label $t(i)$ as finite and set the new pivot $r := i$.
  {If there are more vertices having minimal label $t(\ )$ equal to $t(i)$, only one label can be declared as minimal and chosen as pivot. Other vertices with minimum label $t(\ )$ will be declared as final and chosen as pivots one by one in following steps of computation.}
  GOTO Step 2.

♣

Remark

If we change the stop condition in Step 2. of algorithm as follows:
If all labels $t(\ )$ are final, STOP,
we get a point-to-all version of Dijkstra's algorithm which computes all shortest paths from fixed vertex u to all reachable vertices.



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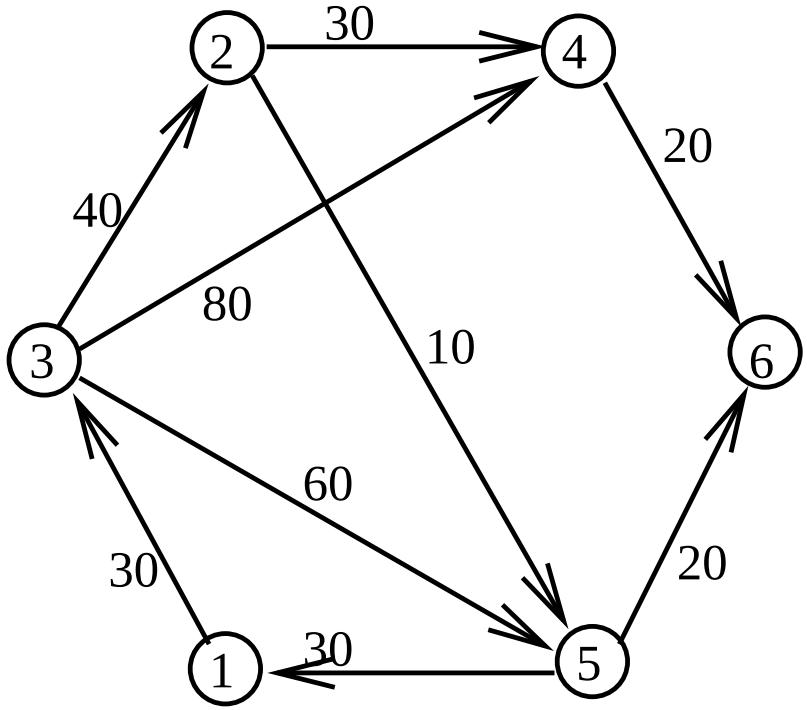
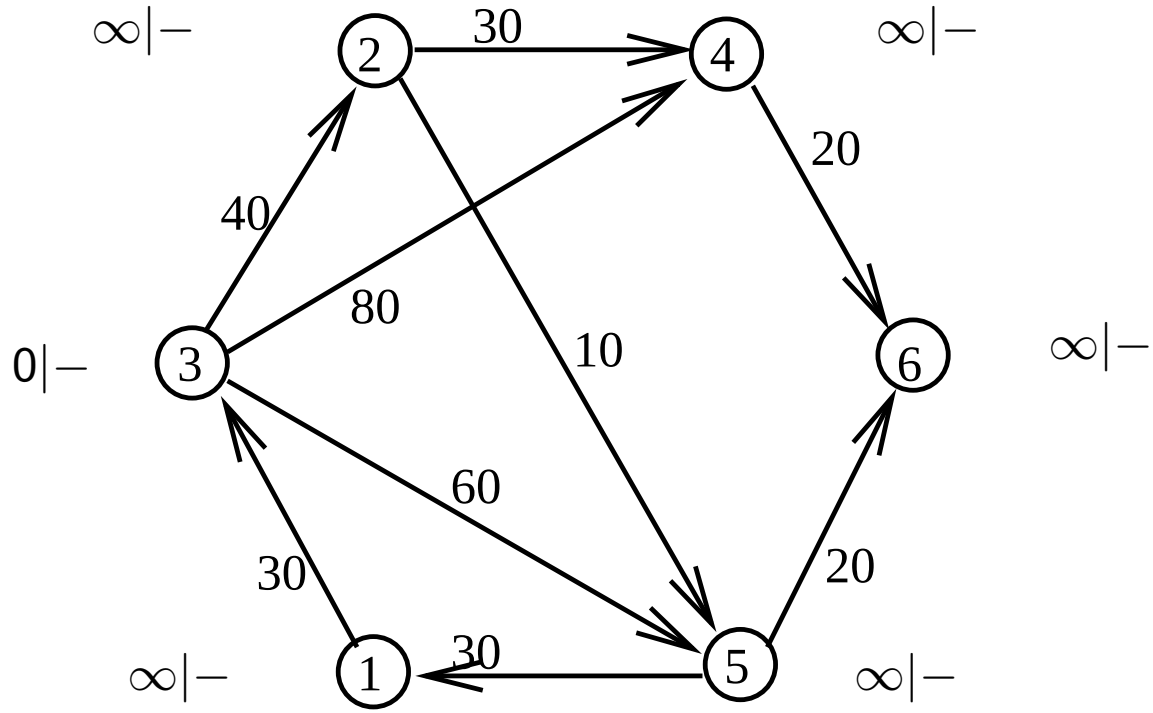
Dijkstra's algorithm – Example



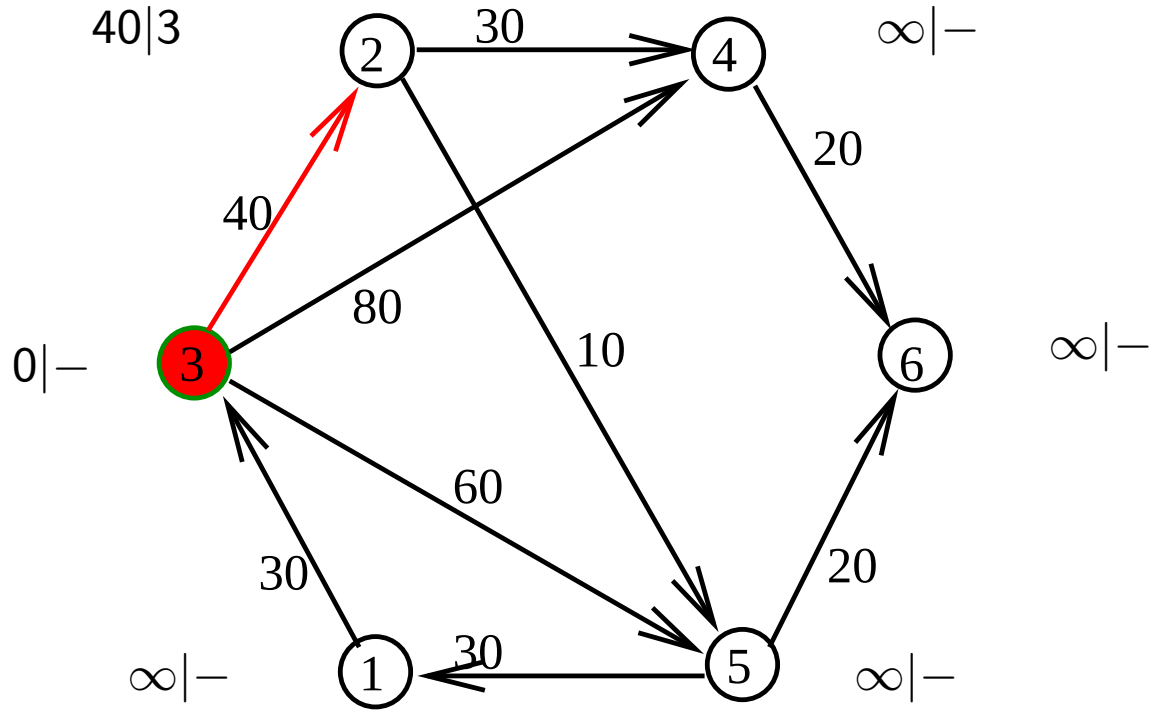
r	$x(r)$	$t(r)$	1	2	3	4	5	6
			$t(v)\|x(v)$					
-	-	-	∞ |	∞ |	**0** |	∞ |	∞ |	∞ |
3	-	0	∞ |	**40 | 3**		80 | 3	60 | 3	∞ |
2	3	40	∞ |			70 | 2	**50 | 2**	∞ |
5	2	50	80 | 5			**70 | 2**		70 | 5
4	2	70	80 | 5					**70 | 5**
6	5	70	**80 | 5**					
1	5	80						

Dijkstra's algorithm – Example



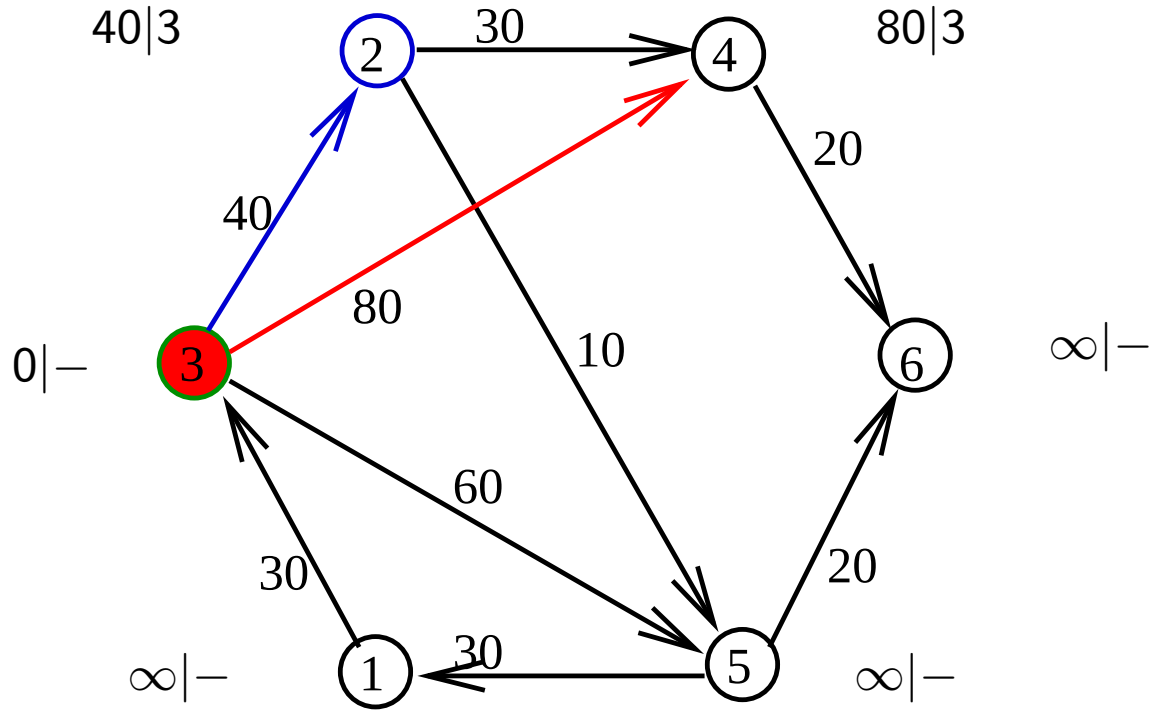
r	$x(r)$	$t(r)$	1	2	3	4	5	6
			$t(v)\|x(v)$					
-	-	-	∞|	∞|	**0**|	∞|	∞|	∞|
3	-	0	∞|	**40|3**		80|3	60|3	∞|
2	3	40	∞|			70|2	**50|2**	∞|
5	2	50	80|5			**70|2**		70|5
4	2	70	80|5					**70|5**
6	5	70	**80|5**					
1	5	80						

Dijkstra's algorithm – Example



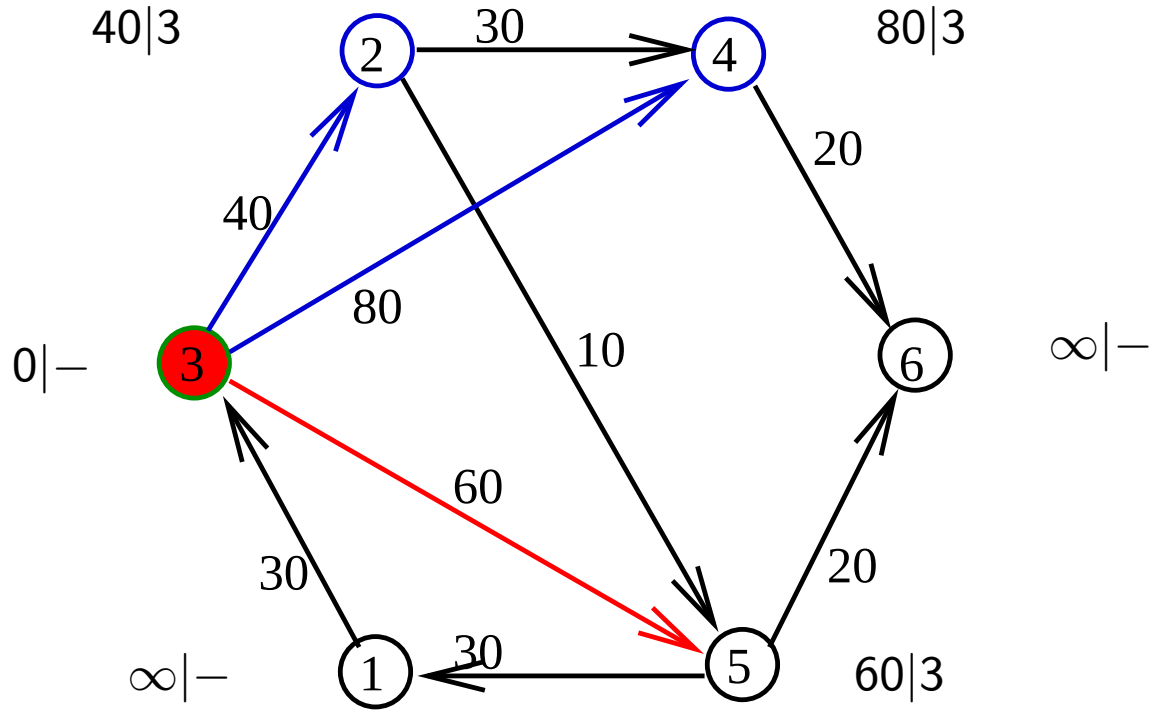
r	$x(r)$	$t(r)$	1	2	3	4	5	6
			$t(v)\|x(v)$					
-	-	-	∞|	∞|	**0**|	∞|	∞|	∞|
3	-	0	∞|	**40|3**		80|3	60|3	∞|
2	3	40	∞|			70|2	**50|2**	∞|
5	2	50	80|5			**70|2**		70|5
4	2	70	80|5					**70|5**
6	5	70	**80|5**					
1	5	80						

Dijkstra's algorithm – Example



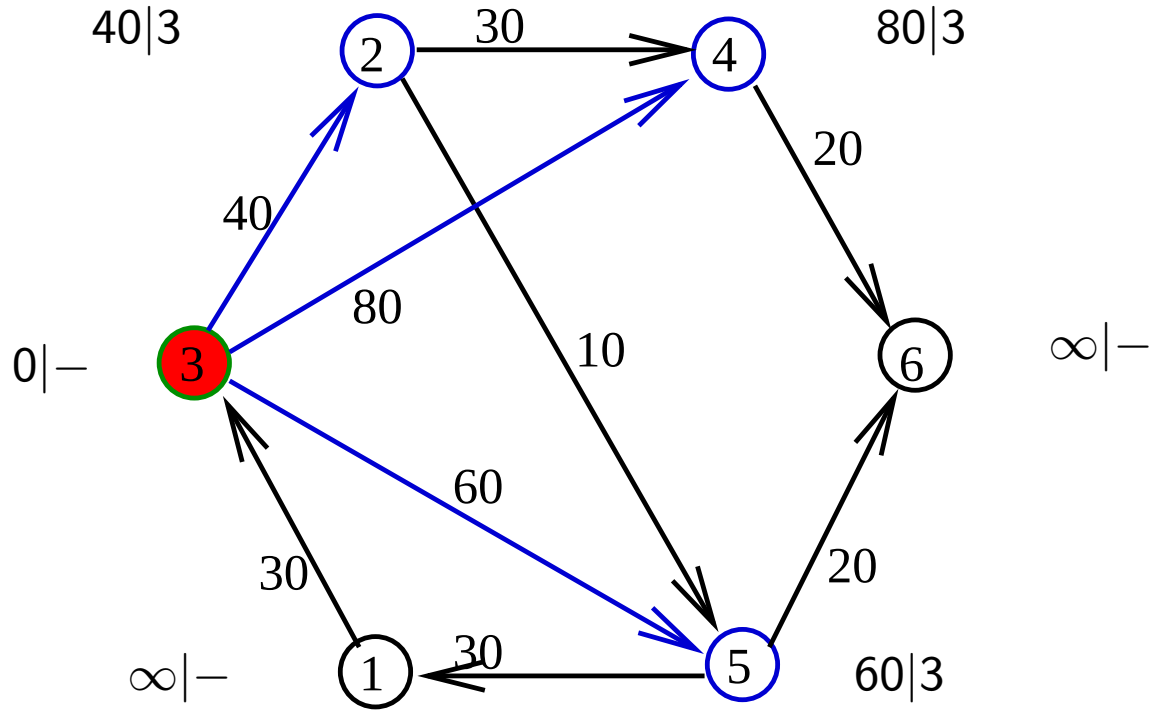
r	$x(r)$	$t(r)$	1	2	3	4	5	6
			$t(v)\|x(v)$					
-	-	-	∞|	∞|	**0**|	∞|	∞|	∞|
3	-	0	∞|	**40|3**		80|3	60|3	∞|
2	3	40	∞|			70|2	**50|2**	∞|
5	2	50	80|5			**70|2**		70|5
4	2	70	80|5					**70|5**
6	5	70	**80|5**					
1	5	80						

Dijkstra's algorithm – Example



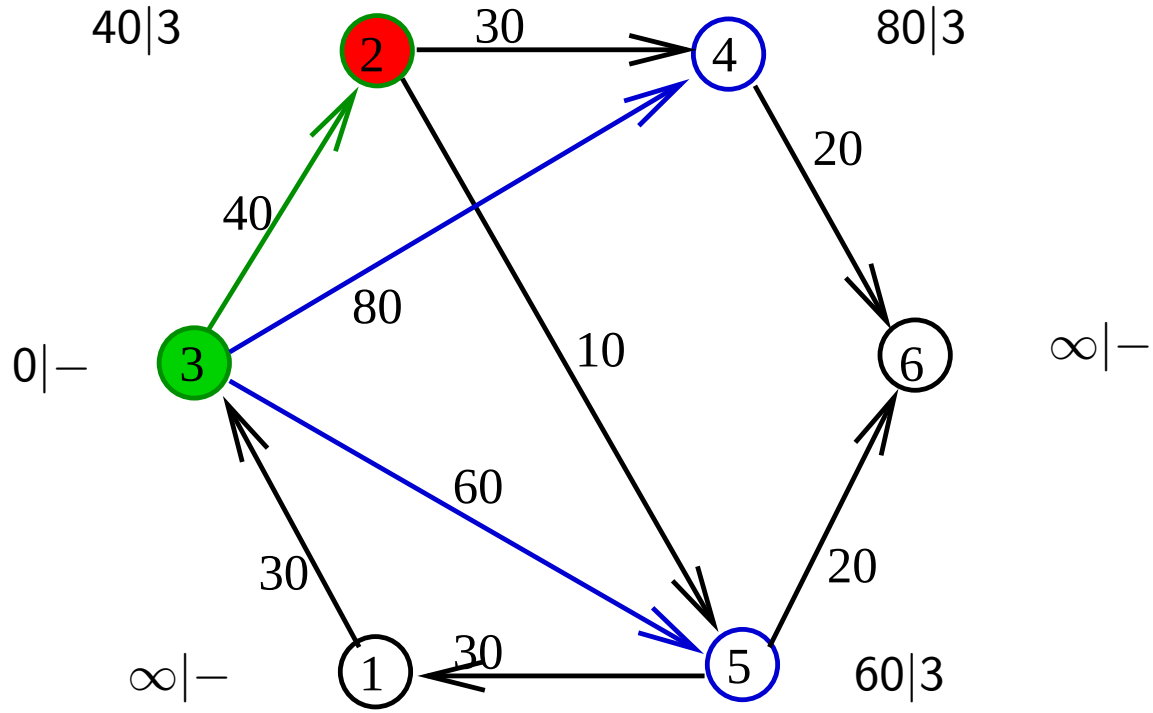
r	$x(r)$	$t(r)$	1	2	3	4	5	6
			$t(v)\|x(v)$					
-	-	-	∞|	∞|	**0**|	∞|	∞|	∞|
3	-	0	∞|	**40|3**		80|3	60|3	∞|
2	3	40	∞|			70|2	**50|2**	∞|
5	2	50	80|5			**70|2**		70|5
4	2	70	80|5					**70|5**
6	5	70	**80|5**					
1	5	80						

Dijkstra's algorithm – Example



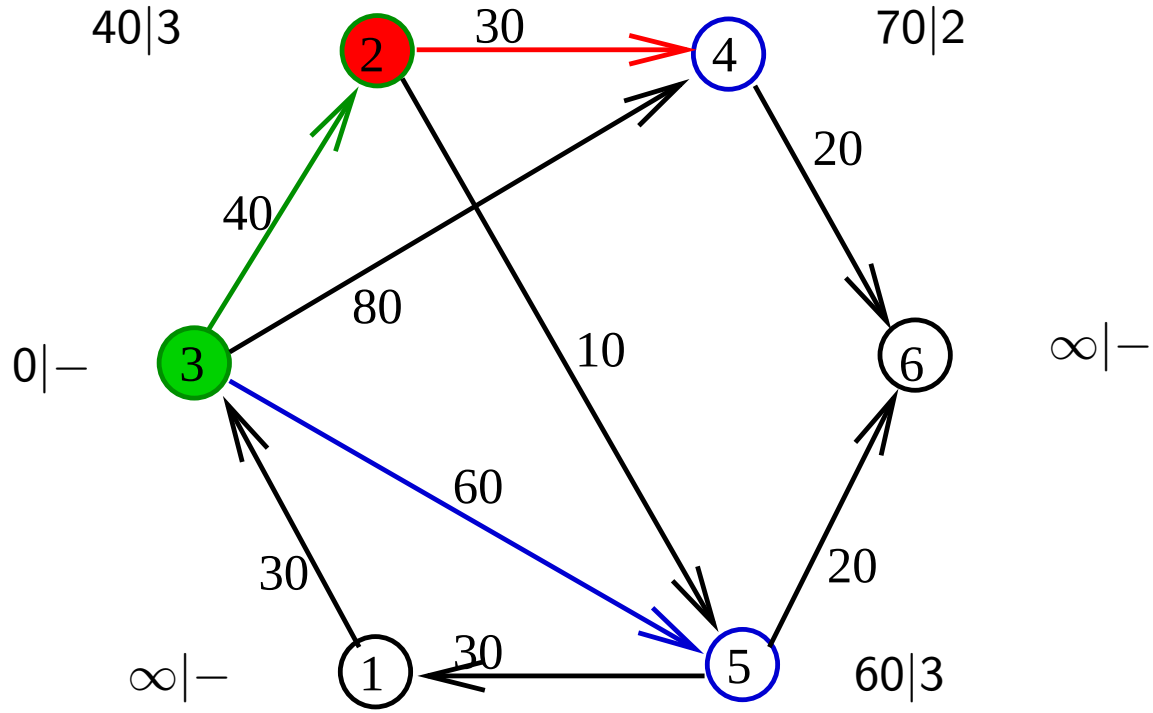
r	$x(r)$	$t(r)$	1	2	3	4	5	6
			$t(v)\|x(v)$					
-	-	-	∞|	∞|	**0**|	∞|	∞|	∞|
3	-	0	∞|	**40|3**		80|3	60|3	∞|
2	3	40	∞|			70|2	**50|2**	∞|
5	2	50	80|5			**70|2**		70|5
4	2	70	80|5					**70|5**
6	5	70	**80|5**					
1	5	80						

Dijkstra's algorithm – Example



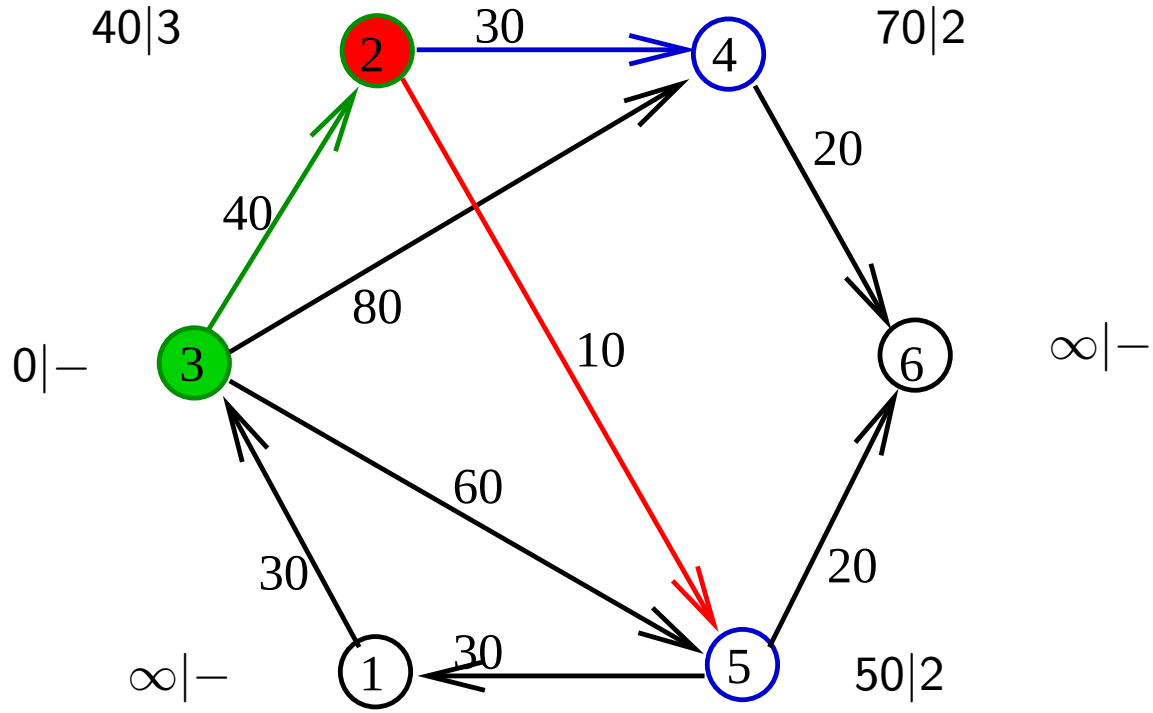
r	$x(r)$	$t(r)$	1	2	3	4	5	6
			$t(v)\|x(v)$					
-	-	-	∞|	∞|	**0**|	∞|	∞|	∞|
3	-	0	∞|	**40|3**		80|3	60|3	∞|
2	3	40	∞|			70|2	**50|2**	∞|
5	2	50	80|5			**70|2**		70|5
4	2	70	80|5					**70|5**
6	5	70	**80|5**					
1	5	80						

Dijkstra's algorithm – Example



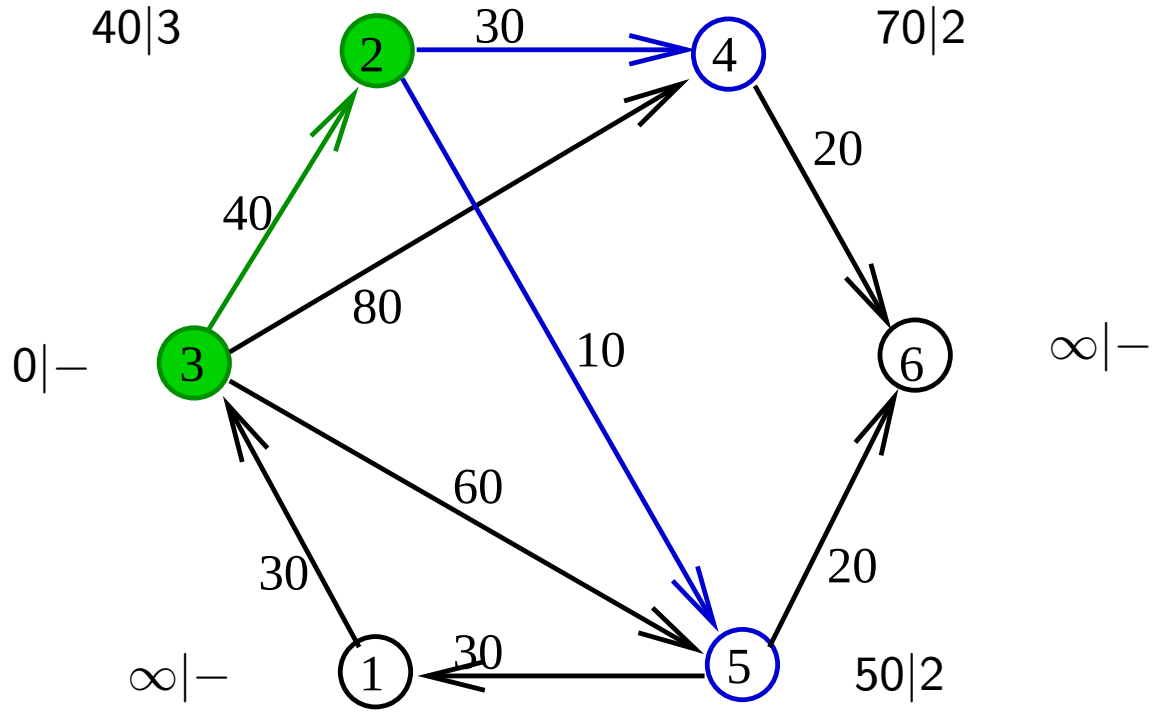
r	$x(r)$	$t(r)$	1	2	3	4	5	6
			$t(v)\|x(v)$					
-	-	-	∞|	∞|	**0**|	∞|	∞|	∞|
3	-	0	∞|	**40**|**3**		80|3	60|3	∞|
2	3	40	∞|			70|2	**50**|**2**	∞|
5	2	50	80|5			**70**|**2**		70|5
4	2	70	80|5					**70**|**5**
6	5	70	**80**|**5**					
1	5	80						

Dijkstra's algorithm – Example



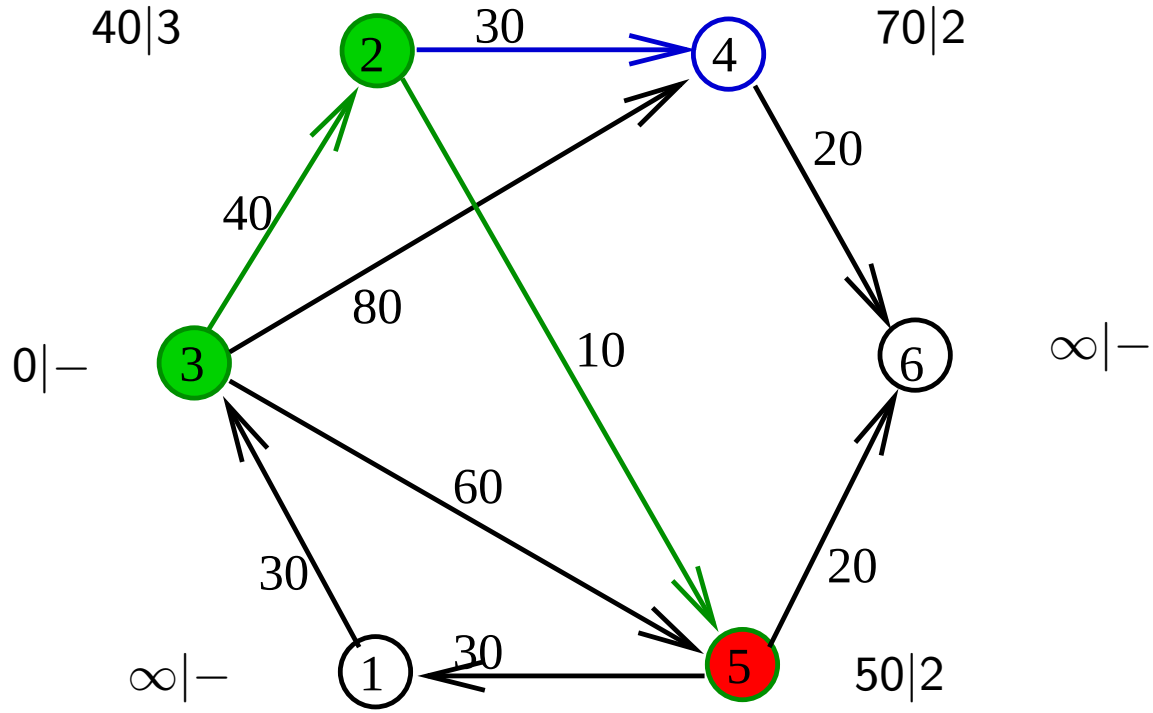
r	$x(r)$	$t(r)$	1	2	3	4	5	6
			$t(v)\|x(v)$					
-	-	-	∞|	∞|	**0**|	∞|	∞|	∞|
3	-	0	∞|	**40|3**		80|3	60|3	∞|
2	3	40	∞|			70|2	**50|2**	∞|
5	2	50	80|5			**70|2**		70|5
4	2	70	80|5					**70|5**
6	5	70	**80|5**					
1	5	80						

Dijkstra's algorithm – Example



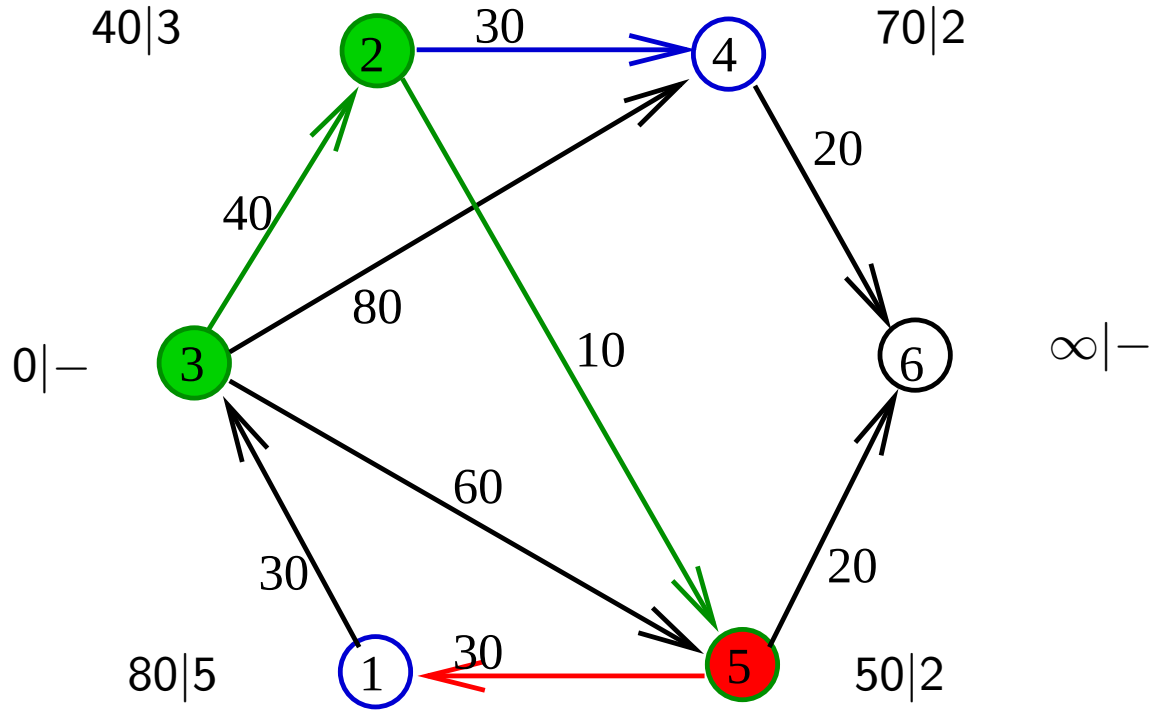
r	$x(r)$	$t(r)$	1	2	3	4	5	6
			$t(v)\|x(v)$					
-	-	-	∞|	∞|	**0**|	∞|	∞|	∞|
3	-	0	∞|	**40|3**		80|3	60|3	∞|
2	3	40	∞|			70|2	**50|2**	∞|
5	2	50	80|5			**70|2**		70|5
4	2	70	80|5					**70|5**
6	5	70	**80|5**					
1	5	80						

Dijkstra's algorithm – Example



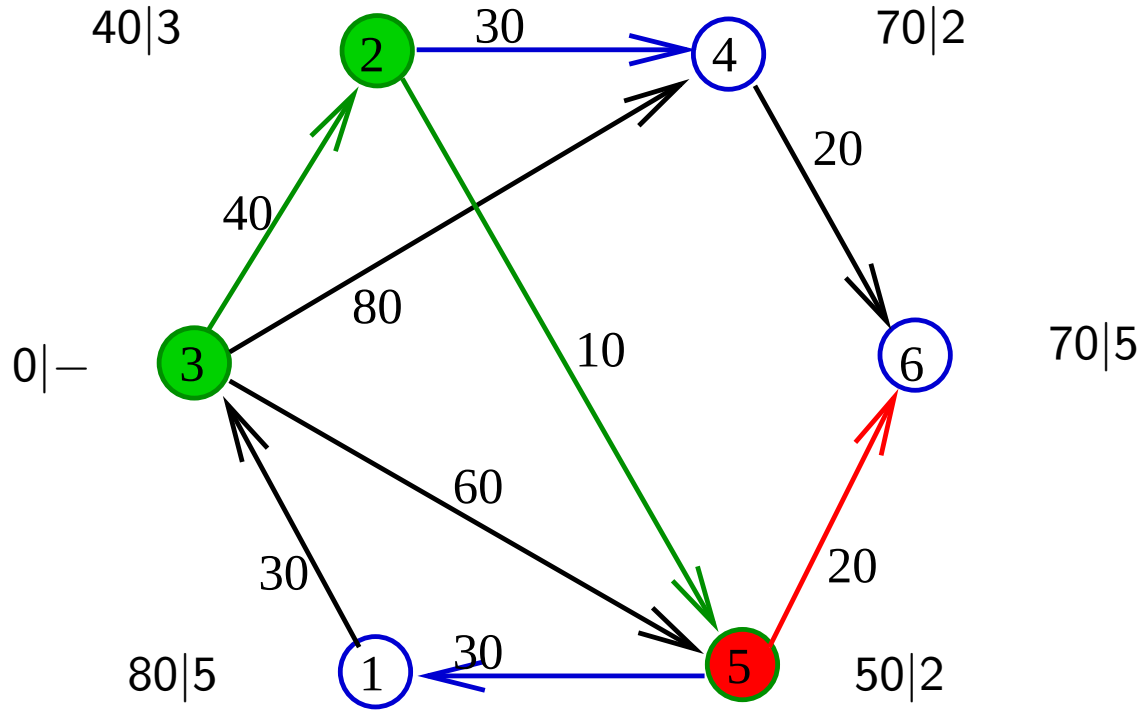
r	$x(r)$	$t(r)$	1	2	3	4	5	6
			$t(v)\|x(v)$					
-	-	-	∞|	∞|	**0**|	∞|	∞|	∞|
3	-	0	∞|	**40**|**3**		80|3	60|3	∞|
2	3	40	∞|			70|2	**50**|**2**	∞|
5	2	50	80|5			**70**|**2**		70|5
4	2	70	80|5					**70**|**5**
6	5	70	**80**|**5**					
1	5	80						

Dijkstra's algorithm – Example



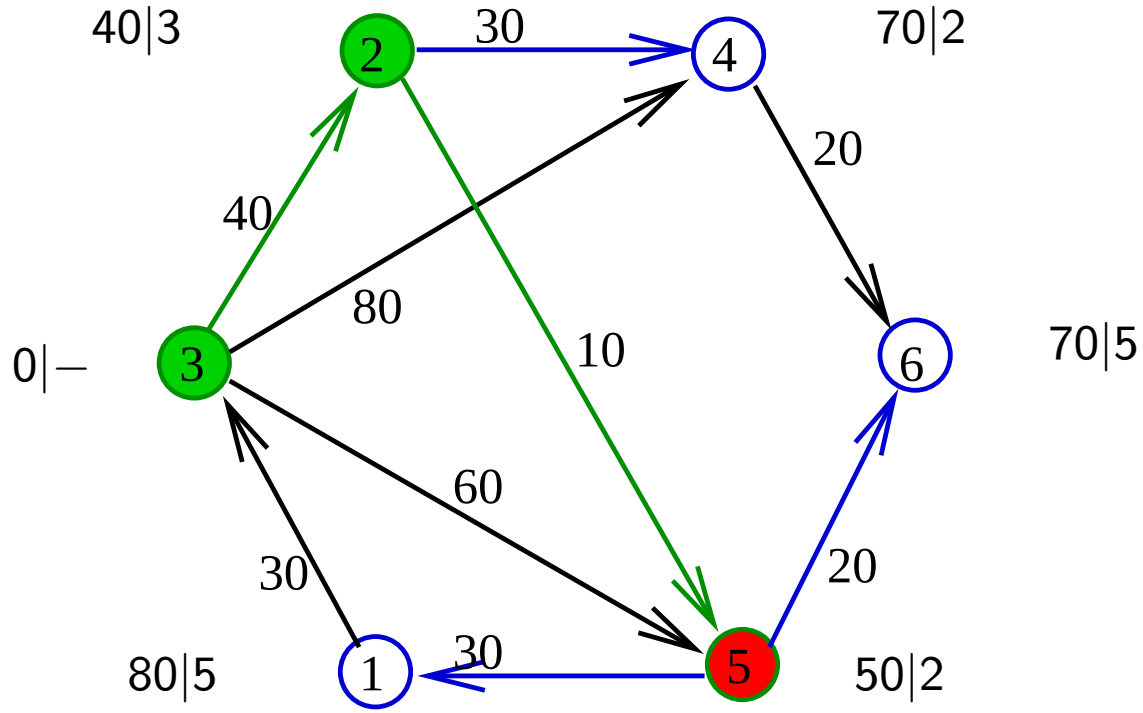
r	$x(r)$	$t(r)$	1	2	3	4	5	6
			$t(v)\|x(v)$					
-	-	-	∞ |	∞ |	**0** |	∞ |	∞ |	∞ |
3	-	0	∞ |	**40|3**		80|3	60|3	∞ |
2	3	40	∞ |			70|2	**50|2**	∞ |
5	2	50	80|5			**70|2**		70|5
4	2	70	80|5					**70|5**
6	5	70	**80|5**					
1	5	80						

Dijkstra's algorithm – Example



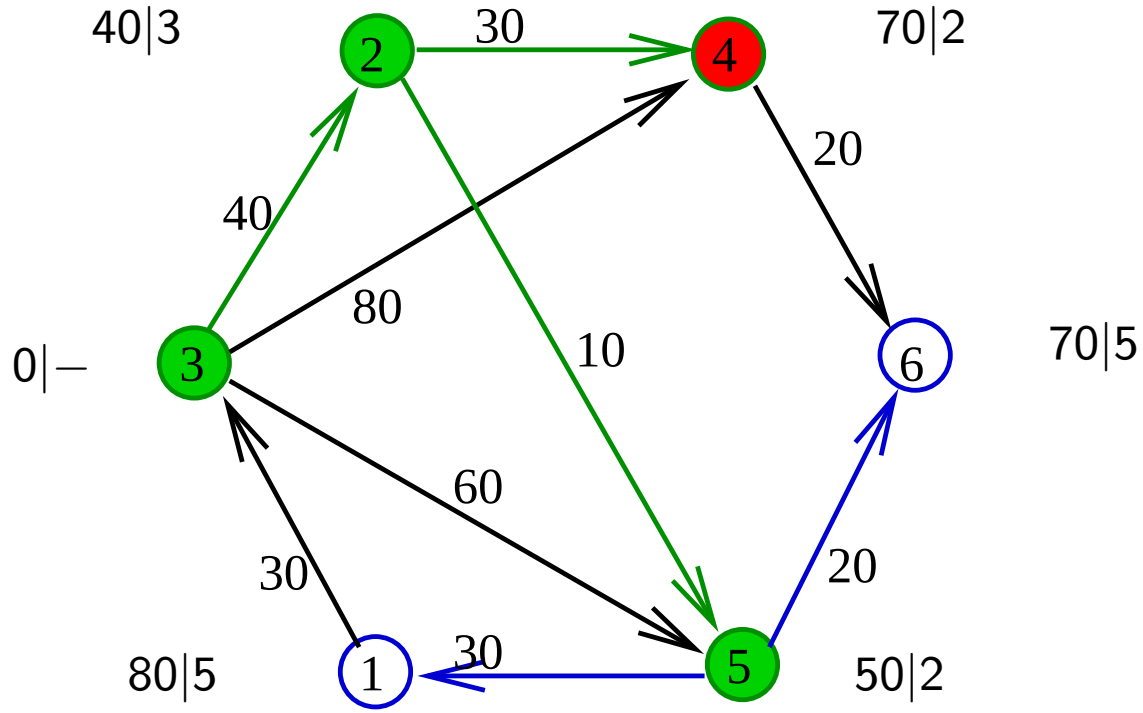
r	$x(r)$	$t(r)$	1	2	3	4	5	6
			$t(v)\|x(v)$					
-	-	-	∞ |	∞ |	**0**|	∞ |	∞ |	∞ |
3	-	0	∞ |	**40**|**3**		80|3	60|3	∞ |
2	3	40	∞ |			70|2	**50**|**2**	∞ |
5	2	50	80|5			**70**|**2**		70|5
4	2	70	80|5					**70**|**5**
6	5	70	**80**|**5**					
1	5	80						

Dijkstra's algorithm – Example



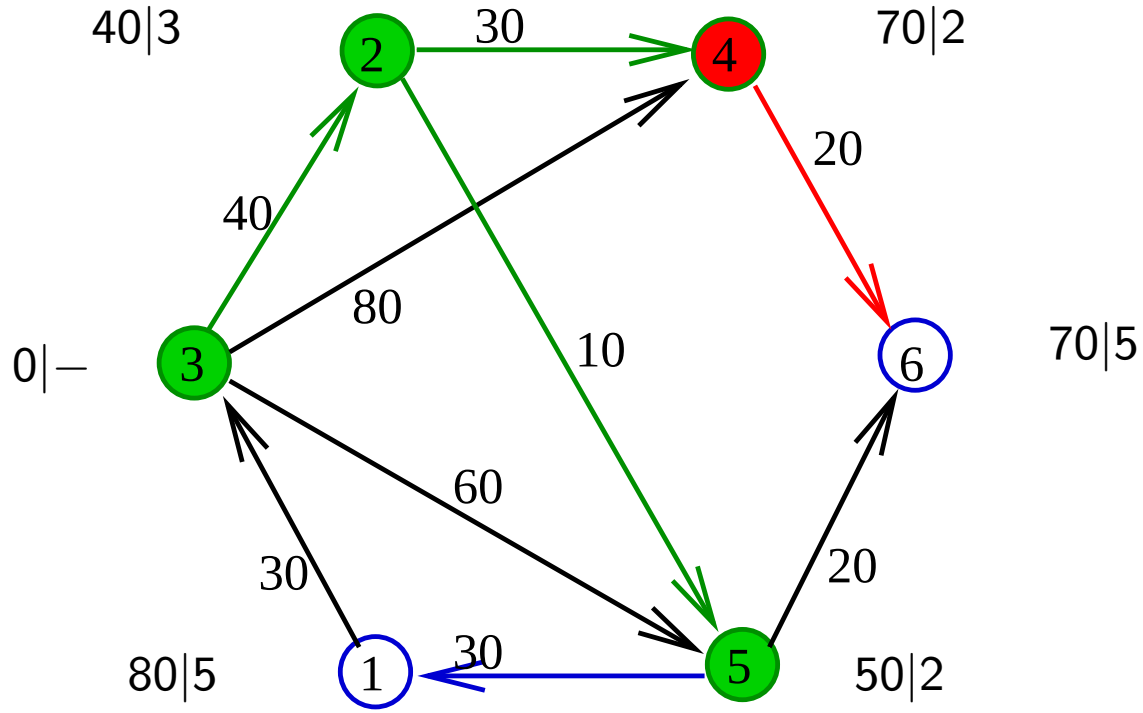
r	$x(r)$	$t(r)$	1	2	3	4	5	6
			$t(v)\|x(v)$					
-	-	-	∞ |	∞ |	**0**|	∞ |	∞ |	∞ |
3	-	0	∞ |	**40**|**3**		80|3	60|3	∞ |
2	3	40	∞ |			70|2	**50**|**2**	∞ |
5	2	50	80|5			**70**|**2**		70|5
4	2	70	80|5					**70**|**5**
6	5	70	**80**|**5**					
1	5	80						

Dijkstra's algorithm – Example



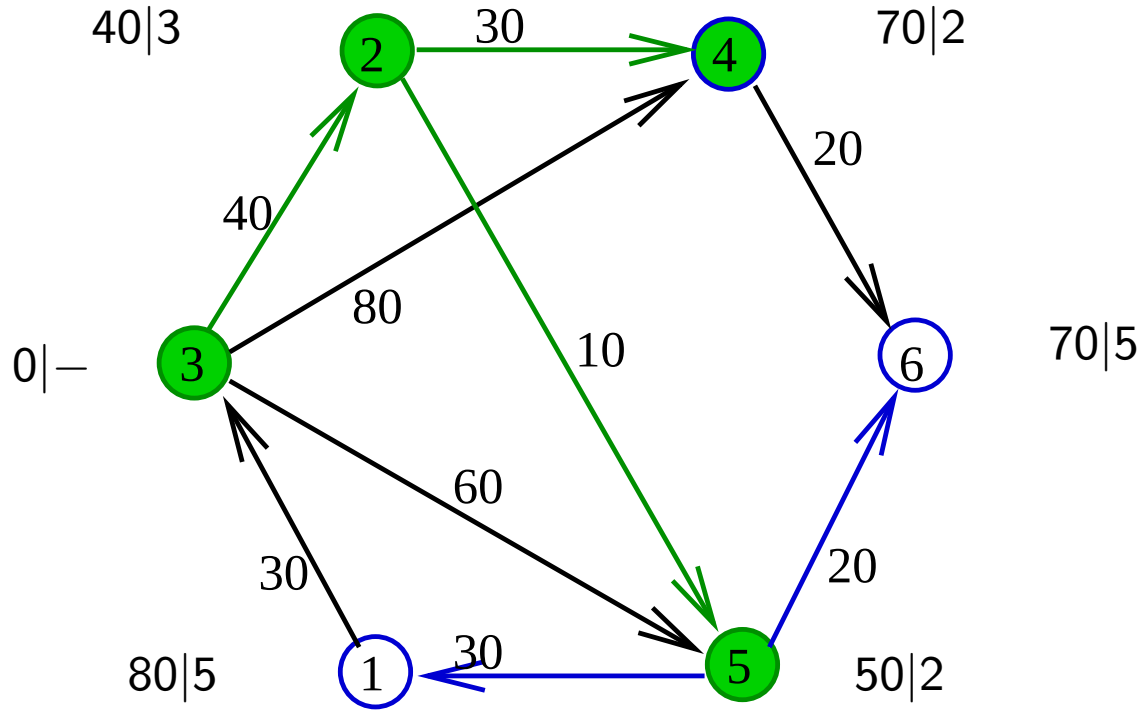
r	$x(r)$	$t(r)$	1	2	3	4	5	6
			$t(v)\|x(v)$					
-	-	-	∞ |	∞ |	**0**|	∞ |	∞ |	∞ |
3	-	0	∞ |	**40**|**3**		80|3	60|3	∞ |
2	3	40	∞ |			70|2	**50**|**2**	∞ |
5	2	50	80|5			**70**|**2**		70|5
4	2	70	80|5					**70**|**5**
6	5	70	**80**|**5**					
1	5	80						

Dijkstra's algorithm – Example



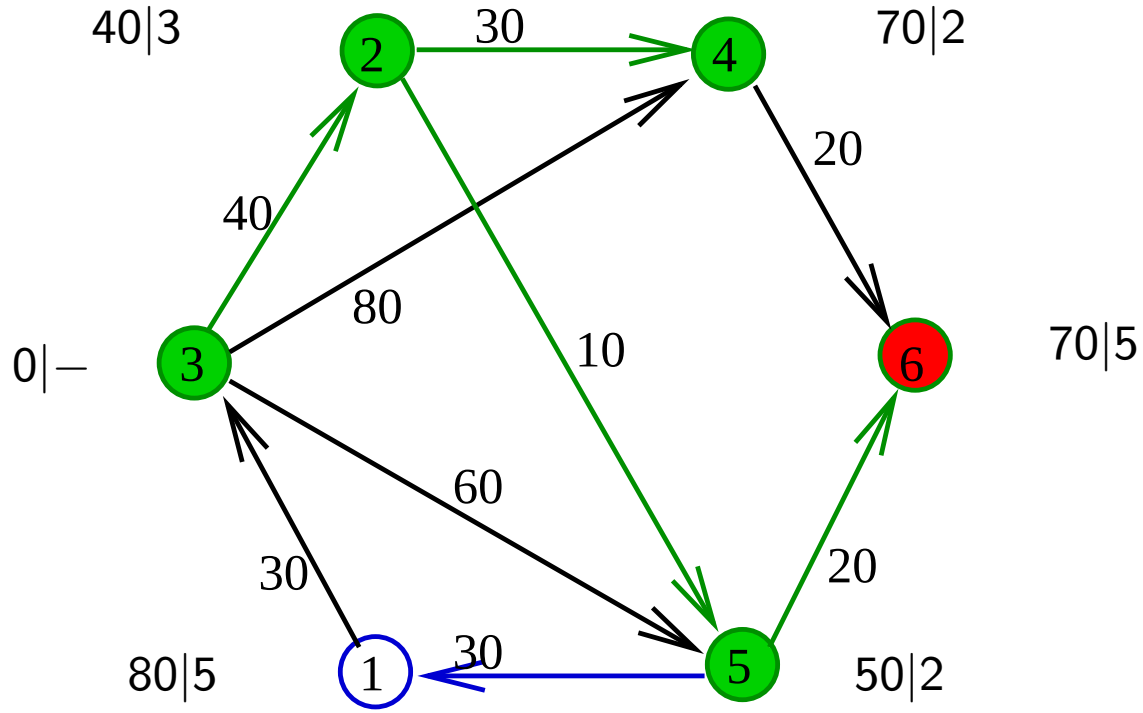
r	$x(r)$	$t(r)$	1	2	3	4	5	6
			$t(v)\|x(v)$					
-	-	-	∞ |	∞ |	**0**|	∞ |	∞ |	∞ |
3	-	0	∞ |	**40**|**3**		80|3	60|3	∞ |
2	3	40	∞ |			70|2	**50**|**2**	∞ |
5	2	50	80|5			**70**|**2**		70|5
4	2	70	80|5					**70**|**5**
6	5	70	**80**|**5**					
1	5	80						

Dijkstra's algorithm – Example



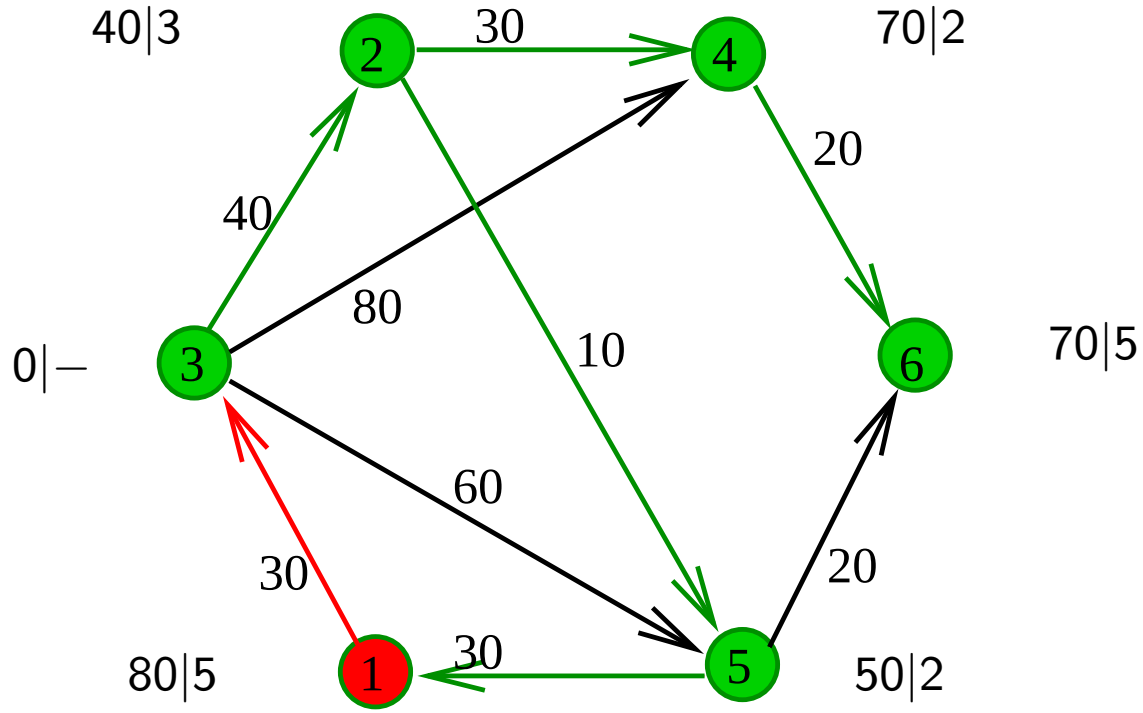
r	$x(r)$	$t(r)$	1	2	3	4	5	6
			$t(v)\|x(v)$					
-	-	-	∞ |	∞ |	**0**|	∞ |	∞ |	∞ |
3	-	0	∞ |	**40**|**3**		80|3	60|3	∞ |
2	3	40	∞ |			70|2	**50**|**2**	∞ |
5	2	50	80|5			**70**|**2**		70|5
4	2	70	80|5					**70**|**5**
6	5	70	**80**|**5**					
1	5	80						

Dijkstra's algorithm – Example



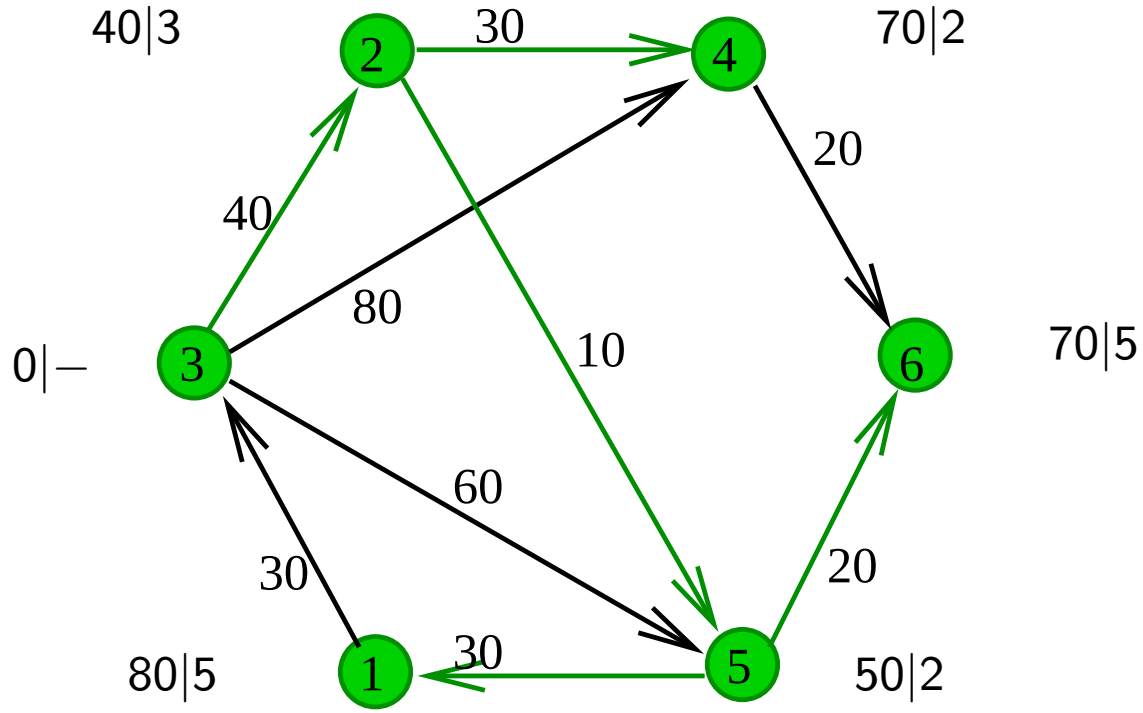
r	$x(r)$	$t(r)$	1	2	3	4	5	6
			$t(v)\|x(v)$					
-	-	-	∞ |	∞ |	**0**|	∞ |	∞ |	∞ |
3	-	0	∞ |	**40|3**		80|3	60|3	∞ |
2	3	40	∞ |			70|2	**50|2**	∞ |
5	2	50	80|5			**70|2**		70|5
4	2	70	80|5					**70|5**
6	5	70	**80|5**					
1	5	80						

Dijkstra's algorithm – Example

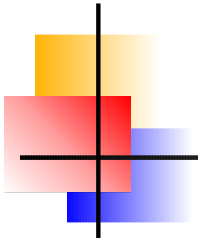


r	$x(r)$	$t(r)$	1	2	3	4	5	6
			$t(v)\|x(v)$					
-	-	-	∞|	∞|	**0**|	∞|	∞|	∞|
3	-	0	∞|	**40**|**3**		80|3	60|3	∞|
2	3	40	∞|			70|2	**50**|**2**	∞|
5	2	50	80|5			**70**|**2**		70|5
4	2	70	80|5					**70**|**5**
6	5	70	**80**|**5**					
1	5	80						

Dijkstra's algorithm – Example



r	$x(r)$	$t(r)$	1	2	3	4	5	6
			$t(v)\|x(v)$					
-	-	-	∞|	∞|	**0**|	∞|	∞|	∞|
3	-	0	∞|	**40|3**		80|3	60|3	∞|
2	3	40	∞|			70|2	**50|2**	∞|
5	2	50	80|5			**70|2**		70|5
4	2	70	80|5					**70|5**
6	5	70	**80|5**					
1	5	80						



Distance matrix

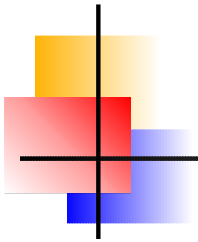
Definition

A real function d defined on cartesian product $V \times V$ i called **a metrics on the set** *V, if it holds:*

1. $d(u, v) \geq 0$ *for every* $u, v \in V$ *and* $d(u, v) = 0$ *if and only if* $u = v$.
2. $d(u, v) = d(v, u)$ *for every* $u, v \in V$.
3. $d(u, w) \leq d(u, v) + d(v, w)$ *for every* $u, v, w \in V$.

Definition

Let $G = (V, H, c)$ be a connected edge weighted graph resp. let $\overrightarrow{G} = (V, H, c)$ be a strongly connected arc weighted digraph, $c(h) > 0$.
The distance of vertices *$u, v \in V$ $d(u, v)$ is the length of the shortest u–v path.*



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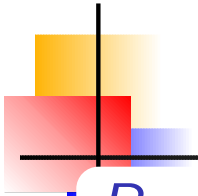

Distance matrix

Remark

Since we have admitted trivial u–u path (containing single vertex u having zero length), the distance $d(u,u) = 0$ for every $u \in V$.

Theorem

If $c(h) > 0$ in a connected graph $G = (V, H, c)$ then distance function $d : V \times V \to \mathbb{R}$ is a metrics on the vertex set V.

Remark

Last theorem does not hold in digraphs – in digraphs need not hold $d(u,v) = d(v,u)$).

Remark

If graph G is disconnected, resp. digraf $\overrightarrow{G}$ is not strongly connected then it is possebl to define $d(u,v) = \infty$ for $u \in V$, $v \in V$ such that v is not reachable from u.

Distance matrix

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Theorem

If $c(h) > 0$ in a connected graph $G = (V, H, c)$ then distance function $d : V \times V \to \mathbb{R}$ is a metrics on the vertex set V.

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Distance matrix

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If $c(h) > 0$ in a connected graph $G = (V, H, c)$ then distance function $d : V \times V \to \mathbb{R}$ is a metrics on the vertex set V.

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Distance matrix

Remark

Since we have admitted trivial u–u path (containing single vertex u having zero length), the distance $d(u, u) = 0$ for every $u \in V$.

Theorem

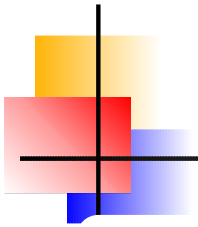
If $c(h) > 0$ in a connected graph $G = (V, H, c)$ then distance function $d : V \times V \to \mathbb{R}$ is a metrics on the vertex set V.

Remark

Last theorem does not hold in digraphs – in digraphs need not hold $d(u, v) = d(v, u)$).

Remark

If graph G is disconnected, resp. digraf $\overrightarrow{G}$ is not strongly connected then it is possebl to define $d(u, v) = \infty$ for $u \in V$, $v \in V$ such that v is not reachable from u.



Radius, center and eccentricity of a graph

Definition

Leth $G = (V, H, c)$ *be an edge weighted connected graph,* $c(h) > 0$. *We define:*

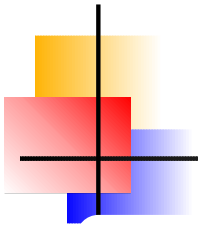
$$\begin{aligned} \textbf{eccentricity of vertex } v \in V \quad & e(v) = \max\{d(u,v) \mid u \in V\} \\ \textbf{radius of graph } G \quad & r(G) = \min\{e(v) \mid v \in V\} \\ \textbf{diameter of graph } G \quad & d(G) = \max\{e(v) \mid v \in V\} \end{aligned}$$

Every vertex of graph G *with minimum eccentricity* $e(v)$ *is called* **a central vertex** *of graph* G, *the set of all central vertices of graph is called* **center of graph** G.

Remark

It can be easily shown that:

$$d(G) = \max\{d(u,v) \mid u \in V, v \in V\}.$$



Radius, center and eccentricity of a graph

Definition

Leth $G = (V, H, c)$ be an edge weighted connected graph, $c(h) > 0$. We define:

$$\begin{aligned} \textbf{eccentricity of vertex } v \in V \quad & e(v) = \max\{d(u,v) \mid u \in V\} \\ \textbf{radius of graph } G \quad & r(G) = \min\{e(v) \mid v \in V\} \\ \textbf{diameter of graph } G \quad & d(G) = \max\{e(v) \mid v \in V\} \end{aligned}$$

Every vertex of graph G with minimum eccentricity $e(v)$ is called **a central vertex** *of graph G, the set of all central vertices of graph is called* **center of graph** *G.*

Remark

It can be easily shown that:

$$d(G) = \max\{d(u,v) \mid u \in V, v \in V\}.$$

Floyd's distance matrix algorithm

Algorithm

Floyd's algorithm *Distance matrix algorithm in an edge weighted graph or digraph* $G = (V, H, c)$, *where* $c(h) \geq 0$.

- **Step 1.** *Create a matrix* $\mathbf{C} = (c_{ij})$ *with entries defined as follows:*

$$c_{ii} = 0 \quad \text{for all} \quad i \in V$$

and for all $i,\ j$ *such that* $i \neq j$

$$c_{ij} = \begin{cases} c(i,j), & \text{if } \{i,j\} \in H, \text{ resp. } (i,j) \in H \\ \infty, & \text{if } \{i,j\} \notin H, \text{ resp. } (i,j) \notin H \end{cases}$$

Create a matrix $\mathbf{X} = (x_{ij})$ *with entries:*

$$x_{ii} = i \quad \text{for all} \quad i \in V$$

and for all $i,\ j$ *such that* $i \neq j$

$$x_{ij} = \begin{cases} i \quad , & \text{if } \{i,j\} \in H, \text{ resp. } (i,j) \in H \\ \infty, & \text{if } \{i,j\} \notin H, \text{ resp. } (i,j) \notin H \end{cases}$$

Floyd's distance matrix algorithm

Algorithm (- continued)

- **Step 2.** *For all* $k = 1, 2, \ldots, n = |V|$*:*
  For all $i \neq k$ *such that* $c_{ik} \neq \infty$*, and for all* $j \neq k$ *such that* $c_{kj} \neq \infty$ *do:*
  If $c_{ij} > c_{ik} + c_{kj}$ *the set:*

$$c_{ij} := c_{ik} + c_{kj}$$

$$x_{ij} := x_{kj}$$

♣

Matrix **C** is the distance matrix of graph, resp. digraph G after finishing Floyd's algorithm.

Matrix **X** contains for every pair i, j last but one vertex of the shortest i–j path after Floyd's algorithm stops.

If we need to find shortest i–j path we can make use of matrix **X** of pointers as follows:
Last but one vertex j_1 of i–j shortest path is $j_1 = x_{ij}$. Last but two wertex from behind is $j_2 = x_{ij_1}$, last but three is $j_3 = x_{ij_2}$ etc. until starting vertex i is reached.

Floyd's distance matrix algorithm

Algorithm (- continued)

- **Step 2.** *For all* $k = 1, 2, \ldots, n = |V|$*:*
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Floyd's distance matrix algorithm

Algorithm (- continued)

- **Step 2.** *For all* $k = 1, 2, \ldots, n = |V|$*:*
  For all $i \neq k$ *such that* $c_{ik} \neq \infty$*, and for all* $j \neq k$ *such that* $c_{kj} \neq \infty$ *do:*
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Floyd's distance matrix algorithm

Algorithm (- continued)

- **Step 2.** *For all* $k = 1, 2, \ldots, n = |V|$*:*
  For all $i \neq k$ *such that* $c_{ik} \neq \infty$*, and for all* $j \neq k$ *such that* $c_{kj} \neq \infty$ *do:*
  If $c_{ij} > c_{ik} + c_{kj}$ *the set:*

$$c_{ij} := c_{ik} + c_{kj}$$

$$x_{ij} := x_{kj}$$

♣

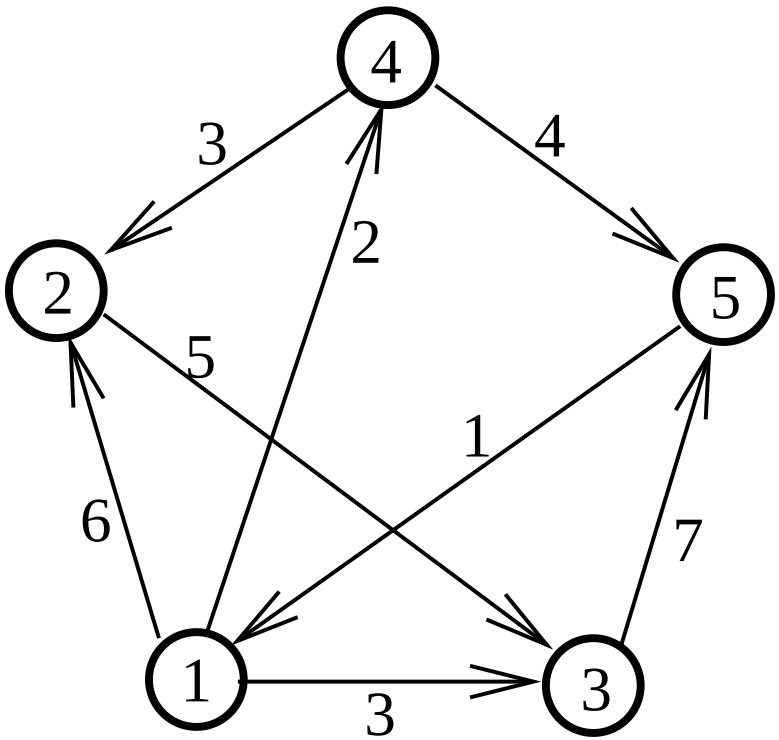
Matrix **C** is the distance matrix of graph, resp. digraph G after finishing Floyd's algorithm.

Matrix **X** contains for every pair i, j last but one vertex of the shortest i–j path after Floyd's algorithm stops.

If we need to find shortest i–j path we can make use of matrix **X** of pointers as follows:
Last but one vertex j_1 of i–j shortest path is $j_1 = x_{ij}$. Last but two wertex from behind is $j_2 = x_{ij_1}$, last but three is $j_3 = x_{ij_2}$ etc. until starting vertex i is reached.

Floyd's distance matrix algorithm – Example

Digraf $\overrightarrow{G} = (V, H, c)$ represented by diagram in the following picture. Our task is to compute corresponding distance matrix using Floyd's algorithm.



Obr.: Digraph $\overrightarrow{G} = (V, H, c)$.

Matrix **C**

	1	2	3	4	5
1	0	6	3	2	∞
2	∞	0	5	∞	∞
3	∞	∞	0	∞	7
4	∞	3	∞	0	4
5	1	∞	∞	∞	0

Matrix **X**

	1	2	3	4	5
1	1	1	1	1	∞
2	∞	2	2	∞	∞
3	∞	∞	3	∞	3
4	∞	4	∞	4	4
5	5	∞	∞	∞	5

Floyd's distance matrix algorithm – Example

Matrix **C**

	1	2	3	4	5
1	**0**	**6**	**3**	**2**	∞
2	∞	0	5	∞	∞
3	∞	∞	0	∞	7
4	∞	3	∞	0	4
5	**1**	∞	∞	∞	0

Matrix **X**

	1	2	3	4	5
1	**1**	**1**	**1**	**1**	∞
2	∞	2	2	∞	∞
3	∞	∞	3	∞	3
4	∞	4	∞	4	4
5	**5**	∞	∞	∞	5

Matrix **C**
after step 2 with $k = 1$

	1	2	3	4	5
1	0	**6**	3	2	∞
2	∞	**0**	**5**	∞	∞
3	∞	∞	0	∞	7
4	∞	**3**	∞	0	4
5	1	**7**	4	3	0

Matrix **X**
after step 2 with $k = 1$

	1	2	3	4	5
1	1	**1**	1	1	∞
2	∞	**2**	**2**	∞	∞
3	∞	∞	3	∞	3
4	∞	**4**	∞	4	4
5	5	**1**	1	1	5

Matrix **C**
after step 2 with $k = 2$

	1	2	3	4	5
1	0	6	**3**	2	∞
2	∞	0	**5**	∞	∞
3	∞	∞	**0**	∞	**7**
4	∞	3	**8**	0	4
5	1	7	**4**	3	0

Matrix **X**
after step 2 with $k = 2$

	1	2	3	4	5
1	1	1	**1**	1	∞
2	∞	2	**2**	∞	∞
3	∞	∞	**3**	∞	**3**
4	∞	4	**2**	4	4
5	5	1	**1**	1	5

Floyd's distance matrix algorithm – Example

Matrix **C**

	1	2	3	4	5
1	**0**	**6**	**3**	**2**	**∞**
2	**∞**	0	5	∞	∞
3	**∞**	∞	0	∞	7
4	**∞**	3	∞	0	4
5	**1**	∞	∞	∞	0

Matrix **C**
after step 2 with $k = 1$

	1	2	3	4	5
1	0	**6**	3	2	∞
2	**∞**	**0**	**5**	**∞**	**∞**
3	∞	**∞**	0	∞	7
4	∞	**3**	∞	0	4
5	1	**7**	4	3	0

Matrix **C**
after step 2 with $k = 2$

	1	2	3	4	5
1	0	6	**3**	2	∞
2	∞	0	**5**	∞	∞
3	∞	∞	**0**	∞	**7**
4	∞	3	**8**	0	4
5	1	7	**4**	3	0

Matrix **X**

	1	2	3	4	5
1	**1**	**1**	**1**	**1**	**∞**
2	**∞**	2	2	∞	∞
3	**∞**	∞	3	∞	3
4	**∞**	4	∞	4	4
5	**5**	∞	∞	∞	5

Matrix **X**
after step 2 with $k = 1$

	1	2	3	4	5
1	1	**1**	1	1	∞
2	**∞**	**2**	**2**	**∞**	**∞**
3	∞	**∞**	3	∞	3
4	∞	**4**	∞	4	4
5	5	**1**	1	1	5

Matrix **X**
after step 2 with $k = 2$

	1	2	3	4	5
1	1	1	**1**	1	∞
2	∞	2	**2**	∞	∞
3	∞	∞	**3**	∞	**3**
4	∞	4	**2**	4	4
5	5	1	**1**	1	5

Floyd's distance matrix algorithm – Example

Matrix **C**

	1	2	3	4	5
1	**0**	**6**	**3**	**2**	**∞**
2	**∞**	0	5	∞	∞
3	**∞**	∞	0	∞	7
4	**∞**	3	∞	0	4
5	**1**	∞	∞	∞	0

Matrix **C**
after step 2 with $k = 1$

	1	2	3	4	5
1	0	**6**	3	2	∞
2	**∞**	**0**	**5**	**∞**	**∞**
3	∞	∞	0	∞	7
4	∞	**3**	∞	0	4
5	1	**7**	4	3	0

Matrix **C**
after step 2 with $k = 2$

	1	2	**3**	4	5
1	0	6	**3**	2	∞
2	∞	0	**5**	∞	∞
3	**∞**	**∞**	**0**	**∞**	**7**
4	∞	3	**8**	0	4
5	1	7	**4**	3	0

Matrix **X**

	1	2	3	4	5
1	**1**	**1**	**1**	**1**	**∞**
2	**∞**	2	2	∞	∞
3	**∞**	∞	3	∞	3
4	**∞**	4	∞	4	4
5	**5**	∞	∞	∞	5

Matrix **X**
after step 2 with $k = 1$

	1	2	3	4	5
1	1	**1**	1	1	∞
2	**∞**	**2**	**2**	**∞**	**∞**
3	∞	∞	3	∞	3
4	∞	**4**	∞	4	4
5	5	**1**	1	1	5

Matrix **X**
after step 2 with $k = 2$

	1	2	**3**	4	5
1	1	1	**1**	1	∞
2	∞	2	**2**	∞	∞
3	**∞**	**∞**	**3**	**∞**	**3**
4	∞	4	**2**	4	4
5	5	1	**1**	1	5

Floyd's distance matrix algorithm – Example

Matrix **C**
after step 2 with $k = 3$

	1	2	3	4	5
1	0	6	3	2	10
2	∞	0	5	∞	12
3	∞	∞	0	∞	7
4	∞	3	8	0	4
5	1	7	4	3	0

Matrix **X**
after step 2 with $k = 3$

	1	2	3	4	5
1	1	1	1	1	3
2	∞	2	2	∞	3
3	∞	∞	3	∞	3
4	∞	4	2	4	4
5	5	1	1	1	5

Matrix **C**
after step 2 with $k = 4$

	1	2	3	4	5
1	0	5	3	2	6
2	∞	0	5	∞	12
3	∞	∞	0	∞	7
4	∞	3	8	0	4
5	1	6	4	3	0

Matrix **X**
after step 2 with $k = 4$

	1	2	3	4	5
1	1	4	1	1	4
2	∞	2	2	∞	3
3	∞	∞	3	∞	3
4	∞	4	4	4	4
5	5	4	1	1	5

Matrix **C**
after step 2 with $k = 5$

	1	2	3	4	5
1	0	5	3	2	6
2	13	0	5	15	12
3	6	13	0	10	7
4	5	3	8	0	4
5	1	6	4	3	0

Matrix **X**
after step 2 with $k = 5$

	1	2	3	4	5
1	1	4	1	1	4
2	5	2	2	1	3
3	5	4	3	1	3
4	5	4	2	4	4
5	5	4	1	1	5

Floyd's distance matrix algorithm – Example

Matrix **C**
after step 2 with $k = 3$

	1	2	3	4	5
1	0	6	3	2	10
2	∞	0	5	∞	12
3	∞	∞	0	∞	7
4	∞	3	8	0	4
5	1	7	4	3	0

Matrix **X**
after step 2 with $k = 3$

	1	2	3	4	5
1	1	1	1	1	3
2	∞	2	2	∞	3
3	∞	∞	3	∞	3
4	∞	4	2	4	4
5	5	1	1	1	5

Matrix **C**
after step 2 with $k = 4$

	1	2	3	4	5
1	0	5	3	2	6
2	∞	0	5	∞	12
3	∞	∞	0	∞	7
4	∞	3	8	0	4
5	1	6	4	3	0

Matrix **X**
after step 2 with $k = 4$

	1	2	3	4	5
1	1	4	1	1	4
2	∞	2	2	∞	3
3	∞	∞	3	∞	3
4	∞	4	4	4	4
5	5	4	1	1	5

Matrix **C**
after step 2 with $k = 5$

	1	2	3	4	5
1	0	5	3	2	6
2	13	0	5	15	12
3	6	13	0	10	7
4	5	3	8	0	4
5	1	6	4	3	0

Matrix **X**
after step 2 with $k = 5$

	1	2	3	4	5
1	1	4	1	1	4
2	5	2	2	1	3
3	5	4	3	1	3
4	5	4	2	4	4
5	5	4	1	1	5

Floyd's distance matrix algorithm – Example

Matrix **C**
after step 2 with $k = 3$

	1	2	3	4	5
1	0	6	3	**2**	10
2	∞	0	5	∞	12
3	∞	∞	0	∞	7
4	∞	**3**	**8**	**0**	**4**
5	1	7	4	**3**	0

Matrix **X**
after step 2 with $k = 3$

	1	2	3	4	5
1	1	1	1	**1**	3
2	∞	2	2	∞	3
3	∞	∞	3	∞	3
4	∞	**4**	**2**	**4**	**4**
5	5	1	1	**1**	5

Matrix **C**
after step 2 with $k = 4$

	1	2	3	4	**5**
1	0	5	3	2	**6**
2	∞	0	5	∞	**12**
3	∞	∞	0	∞	**7**
4	∞	3	8	0	**4**
5	**1**	**6**	**4**	**3**	**0**

Matrix **X**
after step 2 with $k = 4$

	1	2	3	4	**5**
1	1	4	1	1	**4**
2	∞	2	2	∞	**3**
3	∞	∞	3	∞	**3**
4	∞	4	4	4	**4**
5	**5**	**4**	**1**	**1**	**5**

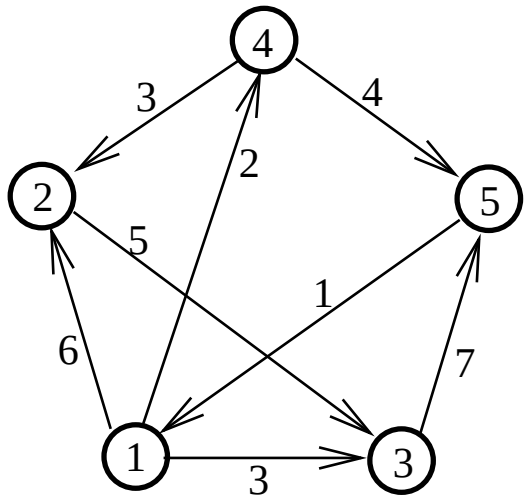
Matrix **C**
after step 2 with $k = 5$

	1	2	3	4	5
1	0	5	3	2	6
2	13	0	5	15	12
3	6	13	0	10	7
4	5	3	8	0	4
5	1	6	4	3	0

Matrix **X**
after step 2 with $k = 5$

	1	2	3	4	5
1	1	4	1	1	4
2	5	2	2	1	3
3	5	4	3	1	3
4	5	4	2	4	4
5	5	4	1	1	5

Floyd's distance matrix algorithm – Example – Results



Matrix **C**
after step 2 with $k = 5$

	1	2	3	4	5
1	0	5	3	2	6
2	13	0	5	15	12
3	6	13	0	**10**	7
4	5	3	8	0	4
5	1	6	4	3	0

i-th row of resultin matrix **X** contain pointers for construction all shortest *i*-*j* paths.

Matrix **X**
after step 2 with $k = 5$

	1	2	3	4	5
1	1	4	1	1	4
2	5	2	2	1	3
3	**5**	4	3	**1**	**3**
4	5	4	2	4	4
5	5	4	1	1	5

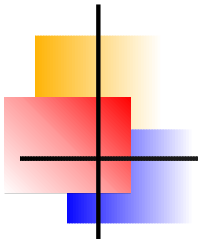
Le us search for the shortest 3-4 paths.

$x_{3,4} = 1$ – last but one vertex of searched path is vertex 1

$x_{3,1} = 5$ – last but two vertex is vertex 5

$x_{3,5} = 3$ – vertex 3 is the starting vertex of searched path

Shortest 3-4 path is $(\mathbf{3}, (3,5), \mathbf{5}, (5,1), \mathbf{1}, (1,4), \mathbf{4})$ and its length is $c_{3,4} = \mathbf{10}$.



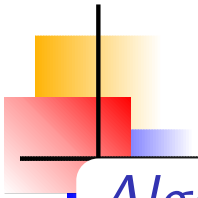
Floyd's distance matrix algorithm – Example – Programm

Example

Floyd's algorithm:

```
for(k=1; k <= n; k++)
   for(i=1; i <= n; i++)
      for(j=1; j <= n; j++)
         if(c[i,j] > c[i,k] + c[k,j])
         {  c[i,j] = c[i,k] + c[k,j];
            x[i,j] = x[k,j]; }
```

The number of elementary operations following statment **if** *in inner cycle* **for***(j=1; ...) (printed in blue) can be limited from above by a fixed number K. This statement will be executed n^3 times. The number of elementry steps of Floyd's algorithm can be limited from above by Kn^3. Hence Floyd's algorithm is an $O(n^3)$ algorithm.*



Label-set a Label-correct implementation

Algorithm

Label-set a Label-correct implemention of point-to all shortest path algorithm *from a fixed starting vertex $u \in V$ into all vertices $v \in V$ in an arc weighted digraph $\overrightarrow{G} = (V, H, c)$ with $c(h) \geq 0$. Suppose $0 \notin V$.*

- **Step 1: Initializtion.**
  Set $t(u) := 0$, $t(i) := \infty$ for $i \in V$, $i \neq u$ and $x(i) := 0$ for all $i \in V$.

  Set $\mathcal{E} := \{u\}$.

- **Step 2:** *Extract a vertex $r \in \mathcal{E}$, set $\mathcal{E} := \mathcal{E} - \{r\}$.*
  For all arcs (r, j) such that $(r, j) \in H^+(r)$ do:
  If $t(j) > t(r) + c(r, j)$ then

  $$t(j) := t(r) + c(r, j),\ x(j) := r,\ \mathcal{E} := \mathcal{E} \cup \{j\}.$$

- **Step 3:** *If $\mathcal{E} \neq \emptyset$ GOTO Step 2.*

  If $\mathcal{E} = \emptyset$ then $t(i)$ is equal to the length of the shortest u–i path for every vertex i reachable from u.

  Corresponding shortest u–i path can be created making use of pointers $x(\,)$ in the same way as in prior shortest path algorithms.

Label-set a Label-correct implementation

Algorithm

Label-set a Label-correct implemention of point-to all shortest path algorithm *from a fixed starting vertex $u \in V$ into all vertices $v \in V$ in an arc weighted digraph $\overrightarrow{G} = (V, H, c)$ with $c(h) \geq 0$. Suppose $0 \notin V$.*

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  For all arcs (r, j) such that $(r, j) \in H^+(r)$ do:
  If $t(j) > t(r) + c(r, j)$ then

  $$t(j) := t(r) + c(r, j),\ x(j) := r,\ \mathcal{E} := \mathcal{E} \cup \{j\}.$$

- **Step 3:** *If $\mathcal{E} \neq \emptyset$ GOTO Step 2.*

  If $\mathcal{E} = \emptyset$ then $t(i)$ is equal to the length of the shortest u–i path for every vertex i reachable from u.

  Corresponding shortest u–i path can be created making use of pointers $x(\,)$ in the same way as in prior shortest path algorithms.

Algorithm

Label-set a Label-correct implemention of point-to all shortest path algorithm *from a fixed starting vertex $u \in V$ into all vertices $v \in V$ in an arc weighted digraph $\overrightarrow{G} = (V, H, c)$ with $c(h) \geq 0$. Suppose $0 \notin V$.*

- **Step 1: Initializtion.**
  Set $t(u) := 0$, $t(i) := \infty$ for $i \in V$, $i \neq u$ and $x(i) := 0$ for all $i \in V$.

  Set $\mathcal{E} := \{u\}$.

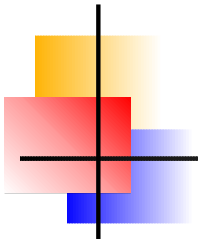
- **Step 2:** *Extract a vertex $r \in \mathcal{E}$, set $\mathcal{E} := \mathcal{E} - \{r\}$.*
  For all arcs (r, j) such that $(r, j) \in H^+(r)$ do:
  If $t(j) > t(r) + c(r, j)$ then

  $$t(j) := t(r) + c(r, j),\ x(j) := r,\ \mathcal{E} := \mathcal{E} \cup \{j\}.$$

- **Step 3:** *If $\mathcal{E} \neq \emptyset$ GOTO Step 2.*

  If $\mathcal{E} = \emptyset$ then $t(i)$ is equal to the length of the shortest u–i path for every vertex i reachable from u.

  Corresponding shortest u–i path can be created making use of pointers $x(\ )$ in the same way as in prior shortest path algorithms.



Label-set a Label-correct implementation

If we extract in the second step of last algorithm pivot $r \in \mathcal{E}$ arbitrarily without any rule we get implementation of fundamental algorithm called **label correct algorithm**.

If we extract in the second step of last algorithm pivot $r \in \mathcal{E}$ with the least value of label $t(\ )$ we get implementation of Dijkstra's algorithm called **label set algorithm**.

It is convenient to organize the set $\mathcal{E}$ as priority queue for label set algorithm.

Label-set a Label-correct implementation - Example

h	$(1,3)$	$(1,7)$	$(2,4)$	$(3,2)$	$(3,5)$	$(4,3)$	$(4,6)$	$(5,1)$	$(5,2)$	$(5,6)$	$(6,1)$	$(7,3)$
$c(h)$	20	120	30	10	60	80	10	30	90	150	20	40

	r	$t(r)$	1	2	3	4	5	6	7	
			$t(v)\|x(v)$							the set $\mathcal{E}$
	-		$\infty\|0$	$\infty\|0$	$0\|0$	$\infty\|0$	$\infty\|0$	$\infty\|0$	$\infty\|0$	3
1.	3	0		$10\|3$			$60\|3$			2 5
2.	2	10				$40\|2$				5 4
3.	5	60	$90\|5$					$210\|5$		4 1 6
4.	4	40						$50\|4$		1 6
5.	1	90							$210\|1$	6 7
6.	6	50	$70\|6$							7 1
7.	7	210								1
8.	1	70							$190\|1$	7
9.	7	190								

Label-set a Label-correct implementation - Example

h	$(1,3)$	$(1,7)$	$(2,4)$	$(3,2)$	$(3,5)$	$(4,3)$	$(4,6)$	$(5,1)$	$(5,2)$	$(5,6)$	$(6,1)$	$(7,3)$
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	r	$t(r)$	1	2	3	4	5	6	7	
							$t(v)\|x(v)$			the set $\mathcal{E}$
	-		$\infty\|0$	$\infty\|0$	$0\|0$	$\infty\|0$	$\infty\|0$	$\infty\|0$	$\infty\|0$	3
1.	3	0		$10\|3$			$60\|3$			2 5
2.	2	10				$40\|2$				5 4
3.	5	60	$90\|5$					$210\|5$		4 1 6
4.	4	40						$50\|4$		1 6
5.	1	90							$210\|1$	6 7
6.	6	50	$70\|6$							7 1
7.	7	210								1
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Label-set a Label-correct implementation - Example

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							$t(v)\|x(v)$			the set $\mathcal{E}$
	-		$\infty\|0$	$\infty\|0$	$0\|0$	$\infty\|0$	$\infty\|0$	$\infty\|0$	$\infty\|0$	3
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4.	4	40						$50\|4$		1 6
5.	1	90							$210\|1$	6 7
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Label-set a Label-correct implementation - Example

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	-		$\infty\|0$	$\infty\|0$	$0\|0$	$\infty\|0$	$\infty\|0$	$\infty\|0$	$\infty\|0$	3
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Label-set a Label-correct implementation - Example

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							$t(v)\|x(v)$			the set $\mathcal{E}$
	-		$\infty\|0$	$\infty\|0$	$0\|0$	$\infty\|0$	$\infty\|0$	$\infty\|0$	$\infty\|0$	3
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2.	2	10				$40\|2$				5 4
3.	5	60	$90\|5$					$210\|5$		4 1 6
4.	4	40						$50\|4$		1 6
5.	1	90							$210\|1$	6 7
6.	6	50	$70\|6$							7 1
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Label-set a Label-correct implementation - Example

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4.	4	40						$50\|4$		1 6
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5.	1	90							$210\|1$	6 7
6.	6	50	$70\|6$							7 1
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8.	1	70							$190\|1$	7
9.	7	190								

Cycle with negative cost in a digraph

Algorithm

Algorithm for searching a cycle with negative cost *in an arc weighted digraph* $\overrightarrow{G} = (V, H, c)$ *with general arc cost.*

- **Step 1: Initialization.**
  Set $t(i) := 0$ *and* $x(i) := 0$ *for all* $i \in V$.
  Set $\mathcal{E} := V$.
- **Step 2:** *Extract* $r \in \mathcal{E}$, *set* $\mathcal{E} := \mathcal{E} - \{r\}$.
  For every arc (r, j), $(r, j) \in H^+(r)$ *do:*
  If $t(j) > t(r) + c(r, j)$ *then*
  set $t(j) := t(r) + c(r, j)$, $x(j) := r$, $\mathcal{E} := \mathcal{E} \cup \{j\}$ *and try whether vertex* j *is an element of sequence:*

$$x(j), x(x(j)), x(x(x(j))), \ldots$$

  If yes you have just found a cycle with negative cost containing vertex j. *STOP.*
  If no then GOTO Step 3.
- **Step 3:** *If* $\mathcal{E} \neq \emptyset$ *then GOTO Step 2.*
  If $\mathcal{E} = \emptyset$ *then STOP. Digraph* $\overrightarrow{G}$ *does not contain a cycle with negative cost.*

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  STOP.
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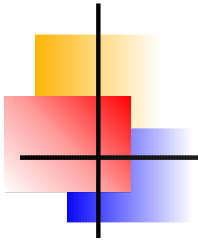


Cycle with negative cost in a digraph

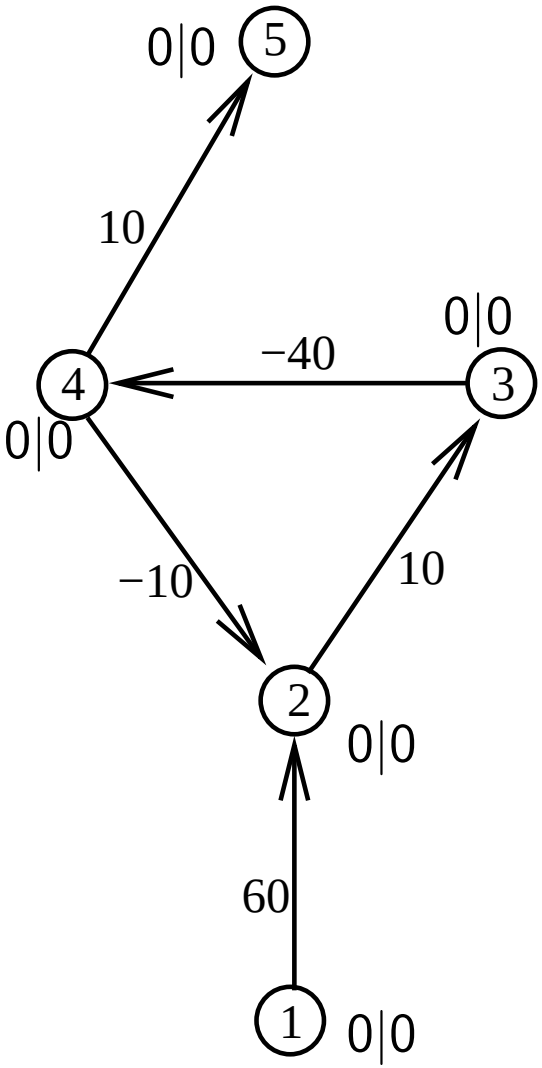
Remark

The sequence

$$x(j), x(x(j)), x(x(x(j))), ...$$

in the second step of last algorithm is necessary to enumerate until it contains vertex j or such vertex $k = x(x(\ldots x(j)\ldots))$ for which it holds $x(k) = 0$.

Cycle with negative cost in a digraph – Example

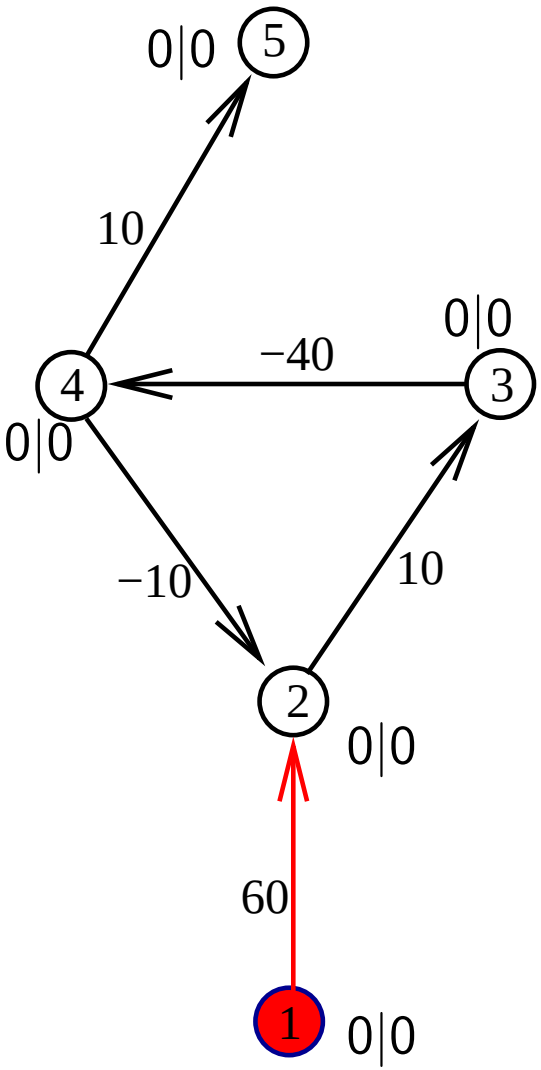


	r	$t(r)$	1	2	3	4	5	
			$t(v)\|x(v)$					the set $\mathcal{E}$
	-		0|0	0|0	0|0	0|0	0|0	1 2 3 4 5
1.	1	0						2 3 4 5
2.	2	0						3 4 5
3.	3	0				−40|3		4 5
4.	4	−40		−50|4			−30|4	5 2
5.	5	0						2
6.	2	−50			−40|2			3

$$x^{(k)}(j) = \underbrace{x(x(\ldots x(}_{k\text{-times}} j)\ldots x(j)\ldots))$$

	j	$x(j)$	$x^{(2)}(j)$	$x^{(3)}(j)$	$x^{(4)}(j)$	$x^{(5)}(j)$
1.	-					
2.	-					
3.	4	3	0			
4.	2	4	3	0		
4.	5	4	3	0		
5.	-					
6.	3	2	4	3		

Cycle with negative cost in a digraph – Example

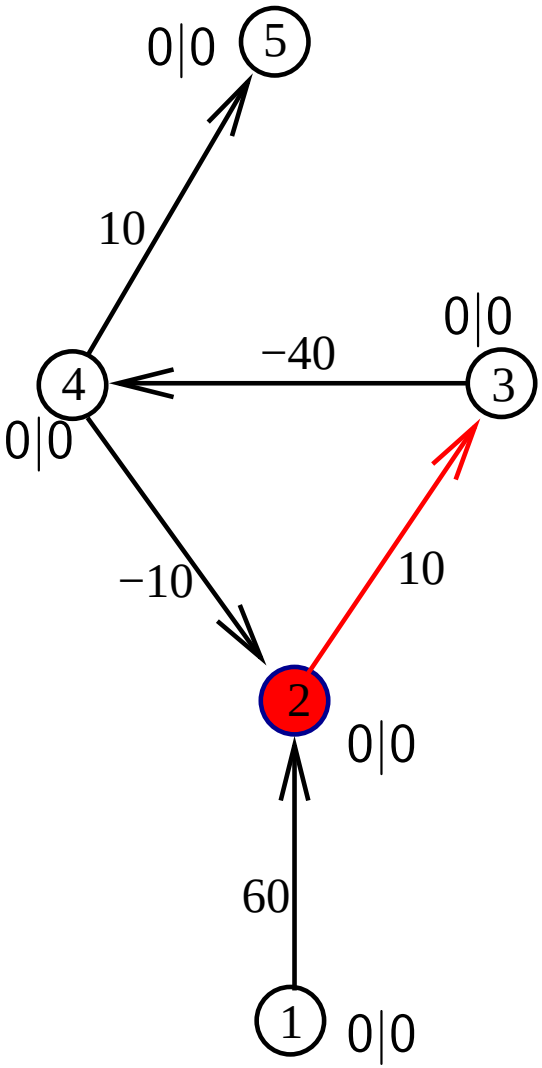


	r	$t(r)$	1	2	3	4	5	
			$t(v)\|x(v)$					the set $\mathcal{E}$
	-		0|0	0|0	0|0	0|0	0|0	1 2 3 4 5
1.	1	0						2 3 4 5
2.	2	0						3 4 5
3.	3	0				−40|3		4 5
4.	4	−40		−50|4			−30|4	5 2
5.	5	0						2
6.	2	−50			−40|2			3

$$x^{(k)}(j) = \underbrace{x(x(\ldots x(}_{k\text{-times}} j)\ldots x(j)\ldots))$$

	j	$x(j)$	$x^{(2)}(j)$	$x^{(3)}(j)$	$x^{(4)}(j)$	$x^{(5)}(j)$
1.	-					
2.	-					
3.	4	3	0			
4.	2	4	3	0		
4.	5	4	3	0		
5.	-					
6.	3	2	4	3		

Cycle with negative cost in a digraph – Example

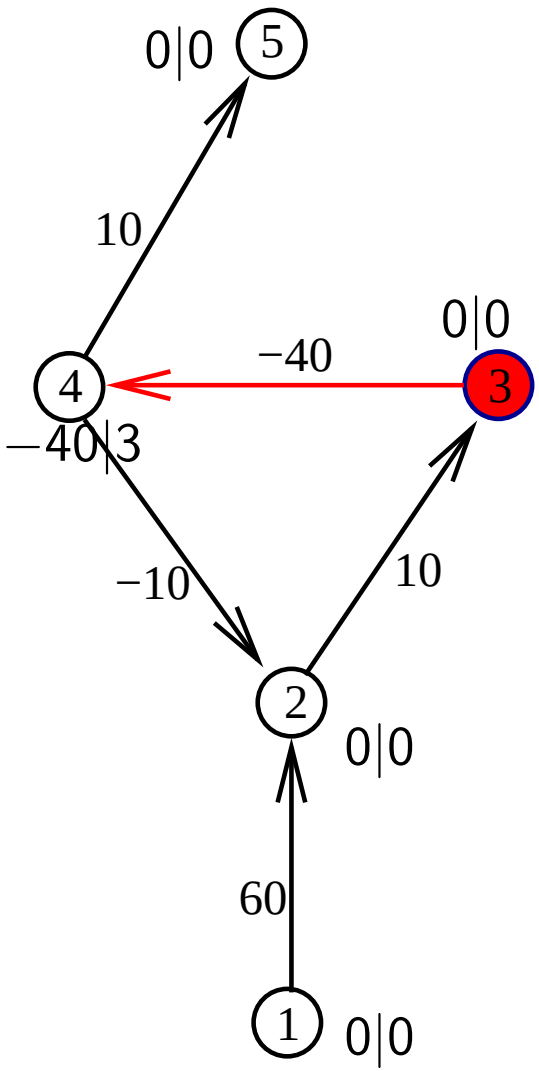


	r	$t(r)$	1	2	3	4	5	
					$t(v)\|x(v)$			the set $\mathcal{E}$
	-		0|0	0|0	0|0	0|0	0|0	1 2 3 4 5
1.	1	0						2 3 4 5
2.	2	0						3 4 5
3.	3	0				−40|3		4 5
4.	4	−40		−50|4			−30|4	5 2
5.	5	0						2
6.	2	−50			−40|2			3

$$x^{(k)}(j) = \underbrace{x(x(\ldots x(}_{k\text{-times}} j) \ldots x(j) \ldots))$$

	j	$x(j)$	$x^{(2)}(j)$	$x^{(3)}(j)$	$x^{(4)}(j)$	$x^{(5)}(j)$
1.	-					
2.	-					
3.	4	3	0			
4.	2	4	3	0		
4.	5	4	3	0		
5.	-					
6.	3	2	4	3		

Cycle with negative cost in a digraph – Example



	r	$t(r)$	1	2	3	4	5	
					$t(v)\|x(v)$			the set $\mathcal{E}$
	-		0|0	0|0	0|0	0|0	0|0	1 2 3 4 5
1.	1	0						2 3 4 5
2.	2	0						3 4 5
3.	3	0				−40|3		4 5
4.	4	−40		−50|4			−30|4	5 2
5.	5	0						2
6.	2	−50			−40|2			3

$$x^{(k)}(j) = \underbrace{x(x(\ldots x(}_{k\text{-times}} j)\ldots x(j)\ldots))$$

	j	$x(j)$	$x^{(2)}(j)$	$x^{(3)}(j)$	$x^{(4)}(j)$	$x^{(5)}(j)$
1.	-					
2.	-					
3.	4	3	0			
4.	2	4	3	0		
4.	5	4	3	0		
5.	-					
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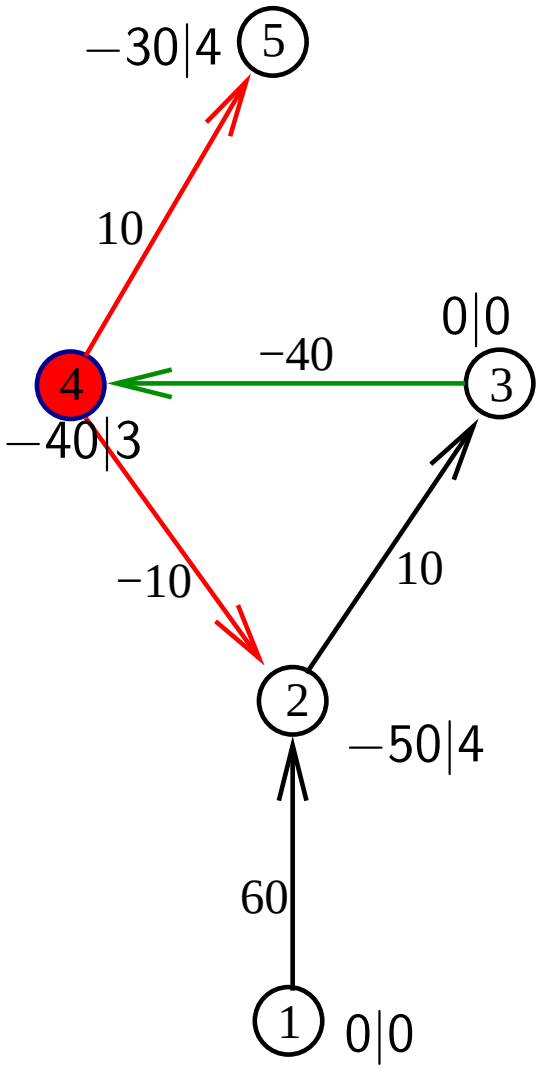
Cycle with negative cost in a digraph – Example

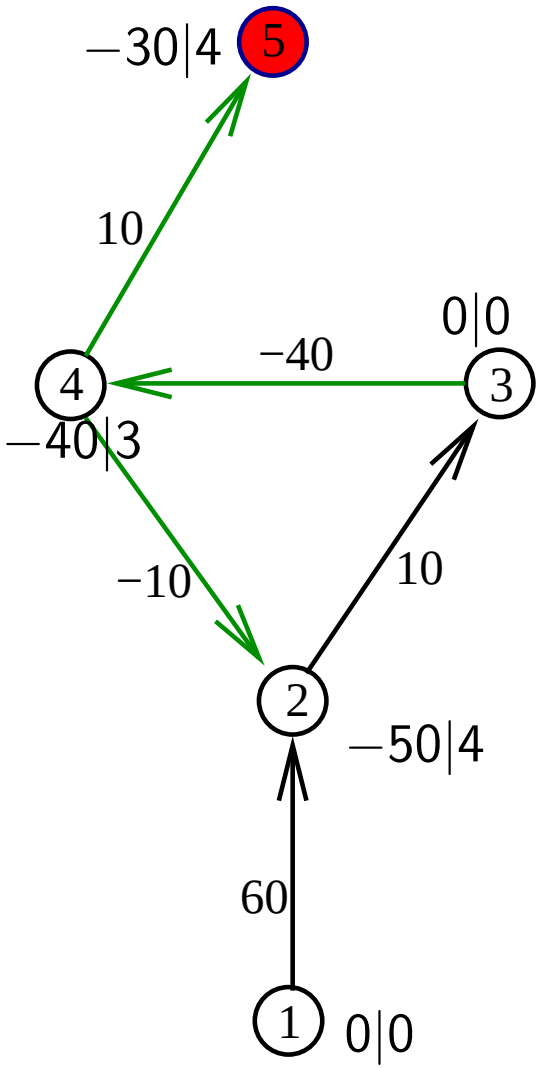


	r	$t(r)$	1	2	3	4	5	
					$t(v)\|x(v)$			the set $\mathcal{E}$
	-		0|0	0|0	0|0	0|0	0|0	1 2 3 4 5
1.	1	0						2 3 4 5
2.	2	0						3 4 5
3.	3	0				−40|3		4 5
4.	4	−40		−50|4			−30|4	5 2
5.	5	0						2
6.	2	−50			−40|2			3

$$x^{(k)}(j) = \underbrace{x(x(\ldots x(}_{k\text{-times}} j)\ldots x(j)\ldots))$$

	j	$x(j)$	$x^{(2)}(j)$	$x^{(3)}(j)$	$x^{(4)}(j)$	$x^{(5)}(j)$
1.	-					
2.	-					
3.	4	3	0			
4.	2	4	3	0		
4.	5	4	3	0		
5.	-					
6.	3	2	4	3		

Cycle with negative cost in a digraph – Example

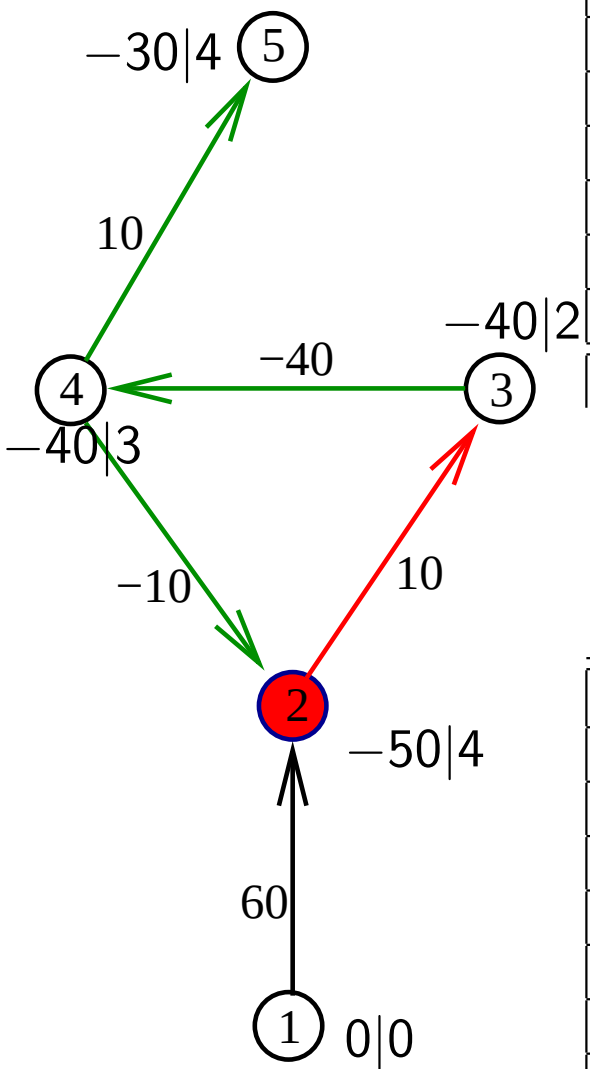


	r	$t(r)$	1	2	3	4	5	
					$t(v)\|x(v)$			the set $\mathcal{E}$
	-		0|0	0|0	0|0	0|0	0|0	1 2 3 4 5
1.	1	0						2 3 4 5
2.	2	0						3 4 5
3.	3	0				−40|3		4 5
4.	4	−40		−50|4			−30|4	5 2
5.	5	0						2
6.	2	−50			−40|2			3

$$x^{(k)}(j) = \underbrace{x(x(\ldots x(}_{k\text{-times}} j) \ldots x(j) \ldots))$$

	j	$x(j)$	$x^{(2)}(j)$	$x^{(3)}(j)$	$x^{(4)}(j)$	$x^{(5)}(j)$
1.	-					
2.	-					
3.	4	3	0			
4.	2	4	3	0		
4.	5	4	3	0		
5.	-					
6.	3	2	4	3		

Cycle with negative cost in a digraph – Example



	r	$t(r)$	1	2	3	4	5	
					$t(v)\|x(v)$			the set $\mathcal{E}$
	-		0|0	0|0	0|0	0|0	0|0	1 2 3 4 5
1.	1	0						2 3 4 5
2.	2	0						3 4 5
3.	3	0				−40|3		4 5
4.	4	−40		−50|4			−30|4	5 2
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$$x^{(k)}(j) = \underbrace{x(x(\ldots x(}_{k\text{-times}} j)\ldots x(j)\ldots))$$

	j	$x(j)$	$x^{(2)}(j)$	$x^{(3)}(j)$	$x^{(4)}(j)$	$x^{(5)}(j)$
1.	-					
2.	-					
3.	4	3	0			
4.	2	4	3	0		
4.	5	4	3	0		
5.	-					
6.	3	2	4	**3**		

Cycle with negative cost in a digraph

Floyd's algorithm 6 (page. 117) can be modified so that it will find a negative cycle in the case of a digraph with general arc weights.
It suffices to define initial matrix **C** in step 1. as follows:

$$c_{ij} = \begin{cases} c(i,j), & \text{ak } \{i,j\} \in H, \quad \text{resp.} \quad (i,j) \in H \\ \infty, & \text{ak } \{i,j\} \notin H, \quad \text{resp.} \quad (i,j) \notin H \end{cases}$$

Matrix **C** contains (by contrast to standard Floyd's algorithm) ∞ on main diagonal.
Matrix **X** stays without change.
Step 2. is the same.
Attention! It is necessary to change elements on main diagonal, too!!!

As soon as a negative number c_{jj} appears on main diagonal we have recovered a negative cycle containing vertex j

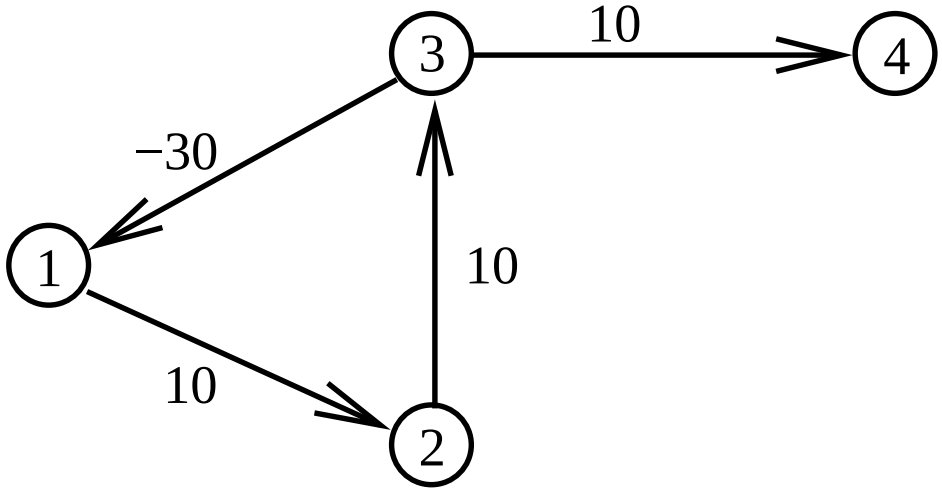
This cycle can be constructed using matrix $\mathbb{X}$.

Example

It is necessary to find out whether digraph represented by following diagram contains a cycle with negative cost.

We write matrices **C** *and* **X** *and execute several times step 2. of Floyd's algorithm for* $k = 1, 2, 3$.

Development of matrices **C** *a* **X** *is illustrated on following tables.*



Obr.: Digraf k príkladu 2.

Príklad

C

∞	10	∞	∞
∞	∞	10	∞
-30	∞	∞	10
∞	∞	∞	∞

C po $k = 1$

∞	10	∞	∞
∞	∞	10	∞
-30	-20	∞	10
∞	∞	∞	∞

C po $k = 2$

∞	10	20	∞
∞	∞	10	∞
-30	-20	**-10**	10
∞	∞	∞	∞

C po $k = 3$

-10	0	20	30
-20	**-10**	10	20
-30	-20	**-10**	10
∞	∞	∞	∞

X

-	1	-	-
-	-	2	-
3	-	-	3
-	-	-	-

X po $k = 1$

-	1	-	-
-	-	2	-
3	1	-	3
-	-	-	-

X po $k = 2$

-	1	2	-
-	-	2	-
3	**1**	**2**	3
-	-	-	-

X po $k = 3$

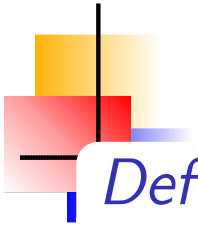
3	1	2	3
3	1	2	3
3	1	2	3
-	-	-	-

After computing third couple of matrices **C**, **X** negative number -10 appears on entry c_{33} what suffices for constatation that examined digraph contains a negative cycle containing vertex 3.

Third row of pointer matrix **X** says that last but one vertex of this cycle is $x_{33} = \mathbf{2}$ last but two vertex is $x_{32} = \mathbf{1}$ and in front of it is vertex $x_{31} = \mathbf{3}$, which is first and also last vertex of cycle.

Searched cycle with negative cost is:

$$3, (3,1), 1, (1,2), 2, (2,3), 3.$$



Path with largest reliability

Definition

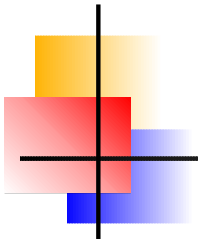
Let $G = (V, H, c)$ be a graph where edge weight $c(h)$ represents the reliability of edge h (reliability – probability of successful travel along the edge h), i.e. $0 \leq c(h) \leq 1$.
Let $\mu(u, v)$ be a u–v path.
The realiability *$s(\mu(u, v))$* **of path** *$\mu(u, v)$ is defined as follows:*

$$s(\mu(u,v)) = \prod_{h \in \mu(u,v)} c(h) \ .$$

u–v u–v **path with maximal probability** *is that u–v path $\mu(u, v)$, which has maximum relibility from all u–v paths.*

Theorem

Let $G = (V, H, c)$ be a graph where $c(h) > 0$ is the reliability of edge $h \in H$.
u–v path $\mu(u, v)$ is the u–v path with maximum reliability in $G = (V, H, c)$ if and only if $\mu(u, v)$ is the shortest path in graph $\overline{G} = (V, H, \overline{c})$ with edge cost $\overline{c}$ fulfilling $\overline{c}(i, j) = -\log_z (c(i, j))$ (where $z > 1$).[a]



Path with largest reliability

Most reliable u-v path

The goal: to maximize

$$s(\mu(u,v)) = \prod_{h \in \mu(u,v)} c(h)$$

$d(\mu(u,v))$ is maximal $\Leftrightarrow$

$\Leftrightarrow \log d(\mu(u,v))$ is maximal.

The goal: to maximize

$$\log s(\mu(u,v)) = \log \prod_{h \in \mu(u,v)} c(h) = \sum_{h \in \mu(u,v)} \log c(h)$$

$\sum_{h \in \mu(u,v)} \log c(h)$ is maximal $\Leftrightarrow$

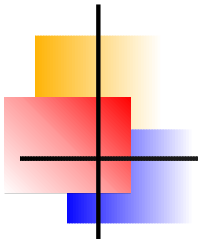
$\Leftrightarrow \sum_{h \in \mu(u,v)} -\log c(h)$ je minimal.

The goal: to minimize $\sum_{h \in \mu(u,v)} -\log c(h)$

The shortest u-v path

The goal: to minimize

$$d(\mu(u,v)) = \sum_{h \in \mu(u,v)} c(h)$$



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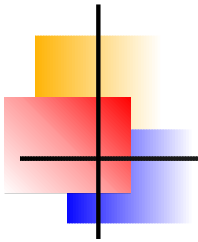

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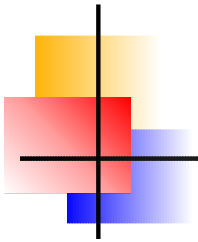

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