Rudolf Blaško

# Integrals and Integrals

2015

# Contents

# 1 Basic integrals

$\int dx = \int 1\,dx = x + c$, for $x \in R$ (i.e. $f(x) = 1$, $x \in R$). [$c$ = constant]

$\int x^a\,dx = \frac{x^{a+1}}{a+1} + c$, for $a \in R$, $a \neq -1$, $x \in R - \{0\}$.

$\int \frac{dx}{x} = \ln|x| + c$, for $x \in R - \{0\}$.

$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$, for $f(x) \neq 0$, $x \in D(f)$.

$\int e^{ax}\,dx = \frac{e^{ax}}{a} + c$, for $a \in R$, $a \neq 0$, $x \in R$, $\int e^x\,dx = e^x + c$.

$\int a^x\,dx = \frac{a^x}{\ln a} + c$, for $a > 0$, $a \neq 1$, $x \in R$, $\int e^x\,dx = \frac{e^x}{\ln e} + c = e^x + c$.

$\int \sin ax\,dx = -\frac{\cos ax}{a} + c$, for $a \in R$, $a \neq 0$, $x \in R$, $\int \sin x\,dx = -\cos x + c$.

$\int \cos ax\,dx = \frac{\sin ax}{a} + c$, for $a \in R$, $a \neq 0$, $x \in R$, $\int \cos x\,dx = \sin x + c$.

$\int \frac{dx}{\cos^2 ax} = \frac{\operatorname{tg} ax}{a} + c$, for $a \in R$, $a \neq 0$, $x \in R - \{(2k+1)\frac{\pi}{2}\,;\, k \in Z\}$, $\int \frac{dx}{\cos^2 x} = \operatorname{tg} x + c$.

$\int \frac{dx}{\sin^2 ax} = -\frac{\operatorname{cotg} ax}{a} + c$, for $a \in R$, $a \neq 0$, $x \in R - \{k\pi\,;\, k \in Z\}$, $\int \frac{dx}{\sin^2 x} = -\operatorname{cotg} x + c$.

$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + c = -\arccos\frac{x}{a} + c$, for $a > 0$, $x \in (-a\,;\, a)$.

$\int \sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\int \frac{dx}{\sqrt{a^2-x^2}}$, for $a > 0$, $x \in (-a\,;\, a)$.

$\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x + \sqrt{x^2-a^2}\right| + c$, for $a > 0$, $x \in (-\infty\,;\, -a) \cup (a\,;\, \infty)$.

$\int \sqrt{x^2-a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\int \frac{dx}{\sqrt{x^2-a^2}}$, for $a > 0$, $x \in (-\infty\,;\, -a) \cup (a\,;\, \infty)$.

$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left(x + \sqrt{x^2+a^2}\right) + c$, for $a > 0$, $x \in R$.

$\int \sqrt{x^2+a^2}\,dx = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\int \frac{dx}{\sqrt{x^2+a^2}}$, for $a > 0$, $x \in R$.

$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + c - \frac{1}{a}\operatorname{arccotg}\frac{x}{a} + c$, for $a > 0$, $x \in R$.

$\int \frac{dx}{x^2-a^2} = \int \frac{1}{2a}\left[\frac{1}{x-a} - \frac{1}{x+a}\right] dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$, for $a > 0$, $x \in R - \{\pm a\}$.

$\int \sinh ax\,dx = \frac{\cosh ax}{a} + c$, for $a \in R$, $a \neq 0$, $x \in R$, $\int \sinh x\,dx = \cosh x + c$.

$\int \cosh ax\,dx = \frac{\sinh ax}{a} + c$, for $a \in R$, $a \neq 0$, $x \in R$, $\int \cosh x\,dx = \sinh x + c$.

$\int \frac{dx}{\cosh^2 ax} = \frac{\operatorname{tgh} ax}{a} + c$, for $a \in R$, $a \neq 0$, $x \in R$, $\int \frac{dx}{\cosh^2 x} = \operatorname{tgh} x + c$.

$\int \frac{dx}{\sinh^2 ax} = -\frac{\operatorname{cotgh} ax}{a} + c$, for $a \in R$, $a \neq 0$, $x \in R - \{0\}$, $\int \frac{dx}{\sinh^2 x} = -\operatorname{cotgh} x + c$.

# 2 Basic integrals — Appendix

$$\int \frac{dx}{\sqrt{a^2-b^2x^2}} = \int \frac{dx}{b\sqrt{\frac{a^2}{b^2}-x^2}} = \frac{1}{b}\arcsin\frac{bx}{a}+c = -\frac{1}{b}\arccos\frac{bx}{a}+c,$$
for $a>0$, $b>0$, $x\in\left(-\frac{a}{b}\,;\,\frac{a}{b}\right)$.

$$\int \frac{dx}{\sqrt{b^2x^2-a^2}} = \int \frac{dx}{b\sqrt{x^2-\frac{a^2}{b^2}}} = \frac{1}{b}\ln\left|x+\sqrt{x^2-\frac{a^2}{b^2}}\right|+c_1 = \frac{1}{b}\ln\left|\frac{bx+\sqrt{b^2x^2-a^2}}{b}\right|+c_1$$
$$= \frac{1}{b}\ln\left|bx+\sqrt{b^2x^2-a^2}\right| - \frac{1}{b}\ln|b|+c_1 = \frac{1}{b}\ln\left|bx+\sqrt{b^2x^2-a^2}\right|+c,$$
for $a>0$, $b>0$, $x\in\left(-\infty\,;\,-\frac{a}{b}\right)\cup\left(\frac{a}{b}\,;\,\infty\right)$.

$$\int \frac{dx}{\sqrt{b^2x^2+a^2}} = \int \frac{dx}{b\sqrt{x^2+\frac{a^2}{b^2}}} = \frac{1}{b}\ln\left(x+\sqrt{x^2+\frac{a^2}{b^2}}\right)+c_1 = \frac{1}{b}\ln\left(\frac{bx+\sqrt{b^2x^2+a^2}}{b}\right)+c_1$$
$$= \frac{1}{b}\ln\left(bx+\sqrt{b^2x^2+a^2}\right) - \frac{1}{b}\ln|b|+c_1 = \frac{1}{b}\ln\left(bx+\sqrt{b^2x^2+a^2}\right)+c,$$
for $a>0$, $b>0$, $x\in R$.

$$\int \frac{dx}{\sqrt{-x^2+qx+r}} = \left[\ -x^2+qx+r = -(x^2-qx-r) = -[(x-\tfrac{q}{2})^2-r-\tfrac{q^2}{4}] = -\frac{(2x-q)^2}{4}+\frac{4r+q^2}{4}\ \right]$$
$$= \arcsin\frac{2x-q}{\sqrt{4r+q^2}}+c = -\arccos\frac{2x-q}{\sqrt{4r+q^2}}+c,$$ for $q,r\in R$, $4r+q^2>0$, $x\in R$.

$$\int \frac{dx}{\sqrt{x^2+qx+r}} = \left[\begin{array}{c} x^2+qx+r = (x+\frac{q}{2})^2+r-\frac{q^2}{4} = \frac{(2x+q)^2}{4}+\frac{4r-q^2}{4} \\ x+\frac{q}{2}+\sqrt{x^2+qx+r} = \frac{2x+q+2\sqrt{x^2+qx+r}}{2} \end{array}\right]$$
$= \ln\left(2x+q+2\sqrt{x^2+qx+r}\right)+c$, for $q,r\in R$, $4r-q^2>0$, $x\in R$,
$= \ln|2x+q|+c$, for $q,r\in R$, $4r-q^2=0$, $x\in R-\left\{-\frac{q}{2}\right\}$,
$= \ln\left|2x+q+2\sqrt{x^2+qx+r}\right|+c$, for $q,r\in R$, $4r-q^2<0$, $x\in R$, $x^2+qx+r>0$.

$$\int \frac{dx}{b^2x^2+a^2} = \frac{1}{b^2}\int \frac{dx}{x^2+\frac{a^2}{b^2}} = \frac{1}{b^2}\frac{b}{a}\operatorname{arctg}\frac{bx}{a}+c = \frac{1}{ab}\operatorname{arctg}\frac{bx}{a}+c$$
$= -\frac{1}{b^2}\frac{b}{a}\operatorname{arccotg}\frac{bx}{a}+c = -\frac{1}{ab}\operatorname{arccotg}\frac{bx}{a}+c$, for $a>0$, $b>0$, $x\in R$.

$$\int \frac{dx}{b^2x^2-a^2} = \frac{1}{b^2}\int \frac{dx}{x^2-\frac{a^2}{b^2}} = \frac{1}{b^2}\frac{b}{2a}\ln\left|\frac{x-\frac{a}{b}}{x+\frac{a}{b}}\right|+c = \frac{1}{2ab}\ln\left|\frac{bx-a}{bx+a}\right|+c,$$
for $a>0$, $b>0$, $x\in R-\left\{\pm\frac{a}{b}\right\}$.

$$\int \frac{dx}{a^2-b^2x^2} = -\int \frac{dx}{b^2x^2-a^2} = -\frac{1}{2ab}\ln\left|\frac{bx-a}{bx+a}\right|+c = \frac{1}{2ab}\ln\left|\frac{bx+a}{bx-a}\right|+c,$$
for $a>0$, $b>0$, $x\in R-\left\{\pm\frac{a}{b}\right\}$.

# 3 Substitution

**Theorem.**

$f(x)\in C(I),\ \ x=\varphi(t),\ \ t\in J,\ \ \varphi(I)\subset I,\ \ \varphi'(t)\in C(J)\ \Longrightarrow$
$$\forall t\in J\colon\ \int f(x)\,\mathrm{d}x=\int f[\varphi(t)]\,\mathrm{d}\varphi(t)=\int f[\varphi(t)]\varphi'(t)\,\mathrm{d}t.$$

---

$$\int\sqrt{1-x^2}\,\mathrm{d}x=\left[\begin{array}{c}x=\sin t,\ t=\arcsin x,\ \mathrm{d}x=\cos t\,\mathrm{d}t,\ x\in(-1\,;\,1),\ t\in\left(-\frac{\pi}{2}\,;\,\frac{\pi}{2}\right)\\ \sqrt{1-x^2}=\sqrt{1-\sin^2 t}=\sqrt{\cos^2 t}=|\cos t|=\cos t\end{array}\right]=\int\cos^2 t\,\mathrm{d}t$$
$$=\int\frac{1+\cos 2t}{2}\,\mathrm{d}t=\frac{t}{2}+\frac{\sin 2t}{4}+c=\frac{t}{2}+\frac{2\sin t\cos t}{4}+c=\frac{\arcsin x}{2}+\frac{x\sqrt{1-x^2}}{2}+c,$$
for $x\in(-1\,;\,1)$.

---

$$\int\sin^3 t\cos t\,\mathrm{d}t=\left[\begin{array}{c}x=\sin t\\ \mathrm{d}x=\cos t\,\mathrm{d}t\end{array}\right]=\int x^3\,\mathrm{d}x=\frac{x^4}{4}+c=\frac{\sin^4 t}{4}+c,$$ for $t\in R$.

---

$$\int\frac{\mathrm{d}x}{\sqrt{x+1}+\sqrt[3]{x+1}}=\int\frac{\mathrm{d}x}{\left(\sqrt[6]{x+1}\right)^3+\left(\sqrt[6]{x+1}\right)^2}=\left[\begin{array}{c}\sqrt[6]{x+1}=t\\ x+1=t^6,\ \mathrm{d}x=6t^5\,\mathrm{d}t\end{array}\right]=\int\frac{6t^5\,\mathrm{d}t}{t^3+t^2}=6\int\frac{t^3\,\mathrm{d}t}{t+1}$$
$$=6\int\frac{t^3+t^2-t^2-t+t+1-1}{t+1}\,\mathrm{d}t=6\int\left[\frac{t^3+t^2}{t+1}-\frac{t^2+t}{t+1}+\frac{t+1}{t+1}-\frac{1}{t+1}\right]\mathrm{d}t$$
$$=6\left[\frac{t^3}{3}-\frac{t^2}{2}+t-\ln|t+1|\right]+c$$
$$=2\left(\sqrt[6]{x+1}\right)^3-3\left(\sqrt[6]{x+1}\right)^2+6\sqrt[6]{x+1}-6\ln\left|1+\sqrt[6]{x+1}\right|+c$$
$$=2\sqrt{x+1}-3\sqrt[3]{x+1}+6\sqrt[6]{x+1}-6\ln\left|1+\sqrt[6]{x+1}\right|+c,$$ for $x>-1$.

---

$$\int\frac{\mathrm{d}x}{\sqrt{5+4\,\mathrm{e}^x}}=\left[\begin{array}{c}\mathrm{e}^x=t,\ t>0\\ x=\ln t,\ \mathrm{d}x=\frac{\mathrm{d}t}{t}\end{array}\right]=\int\frac{\mathrm{d}t}{t\sqrt{5+4t}}=\left[\begin{array}{c}4t=u^2\\ 4\,\mathrm{d}t=2u\,\mathrm{d}u\end{array}\right]=\int\frac{4}{u^2\sqrt{5+u^2}}\frac{u\,\mathrm{d}u}{2}$$
$$=2\int\frac{\mathrm{d}u}{u\sqrt{5+u^2}}=\left[\begin{array}{c}u=\frac{\sqrt{5}}{v},\ \mathrm{d}u=-\frac{\sqrt{5}\,\mathrm{d}v}{v^2},\ u>0,\ v>0\\ \sqrt{5+u^2}=\sqrt{5+\frac{5}{v^2}}=\frac{\sqrt{5}}{v}\sqrt{v^2+1}\end{array}\right]=2\int\frac{v}{\sqrt{5}}\frac{v}{\sqrt{5}\sqrt{v^2+1}}\frac{-\sqrt{5}\,\mathrm{d}v}{v^2}$$
$$=-\frac{2}{\sqrt{5}}\int\frac{\mathrm{d}v}{\sqrt{v^2+1}}=-\frac{2}{\sqrt{5}}\ln\left(v+\sqrt{v^2+1}\right)+c_1=-\frac{2}{\sqrt{5}}\ln\left(\frac{\sqrt{5}}{u}+\sqrt{\frac{5}{u^2}+1}\right)+c_1$$
$$=-\frac{2}{\sqrt{5}}\ln\left(\frac{\sqrt{5}}{u}+\frac{\sqrt{5+u^2}}{u}\right)+c_1=-\frac{2}{\sqrt{5}}\ln\frac{\sqrt{5}+\sqrt{5+u^2}}{u}+c_1$$
$$=\frac{2}{\sqrt{5}}\left[\ln u-\ln\left(\sqrt{5}+\sqrt{5+u^2}\right)\right]+c_1=\frac{2}{\sqrt{5}}\left[\ln(4t)^{\frac{1}{2}}-\ln\left(\sqrt{5}+\sqrt{5+4t}\right)\right]+c_1$$
$$=\frac{2}{\sqrt{5}}\left[\frac{1}{2}\ln(4\,\mathrm{e}^x)-\ln\left(\sqrt{5}+\sqrt{5+4\,\mathrm{e}^x}\right)\right]+c_1$$
$$=\frac{2}{\sqrt{5}}\left[\frac{1}{2}\ln 4+\frac{1}{2}\ln\mathrm{e}^x-\ln\left(\sqrt{5}+\sqrt{5+4\,\mathrm{e}^x}\right)\right]+c_1$$
$$=\frac{\ln 4}{\sqrt{5}}+\frac{x}{\sqrt{5}}-\frac{2}{\sqrt{5}}\ln\left(\sqrt{5}+\sqrt{5+4\,\mathrm{e}^x}\right)+c_1=\frac{x}{\sqrt{5}}-\frac{2}{\sqrt{5}}\ln\left(\sqrt{5}+\sqrt{5+4\,\mathrm{e}^x}\right)+c,$$
for $x\in R$.

---

$$\int f\left(x,\ \sqrt[m]{\tfrac{ax+b}{cx+d}}\right) \mathrm{d}x \quad \textbf{for } ad-bc\neq 0$$

$\left[\text{Substitution } t=\sqrt[m]{\frac{ax+b}{cx+d}}\right] \implies t^m=\frac{ax+b}{cx+d},\quad x=\frac{dt^m-b}{a-ct^m},\quad \mathrm{d}x=\frac{mt^{m-1}(ad-bc)}{(a-ct^m)^2}\,\mathrm{d}t$

$$\int \frac{\sqrt{1-\sqrt{x}}}{\sqrt{1+\sqrt{x}}}\,\mathrm{d}x = \int \frac{\sqrt{1-\sqrt{x}}}{\sqrt{1+\sqrt{x}}}\cdot\frac{\sqrt{1-\sqrt{x}}}{\sqrt{1-\sqrt{x}}}\,\mathrm{d}x = \int \frac{1-\sqrt{x}}{\sqrt{1-x}}\,\mathrm{d}x = \int \frac{\mathrm{d}x}{\sqrt{1-x}} - \int \frac{\sqrt{x}}{\sqrt{1-x}}\,\mathrm{d}x$$
$$= \left[\begin{array}{c|c} 1-x=u & \frac{x}{1-x}=t^2,\ x=t^2-xt^2,\ x=\frac{t^2}{1+t^2} \\ \mathrm{d}x=-\mathrm{d}u & \mathrm{d}x=\frac{2t(1+t^2)-t^2\cdot 2t}{(1+t^2)^2}\,\mathrm{d}t=\frac{2t\,\mathrm{d}t}{(1+t^2)^2}\end{array}\right] = \int (1-x)^{-\frac{1}{2}}\,\mathrm{d}x - \int \sqrt{\frac{x}{1-x}}\,\mathrm{d}x$$
$$= -\int u^{-\frac{1}{2}}\,\mathrm{d}u - \int \frac{t\cdot 2t\,\mathrm{d}t}{(1+t^2)^2} = -\frac{u^{\frac{1}{2}}}{\frac{1}{2}} - 2\int \frac{t^2\,\mathrm{d}t}{(1+t^2)^2} = -2\sqrt{u} - 2\int \frac{1+t^2-1}{(1+t^2)^2}\,\mathrm{d}t$$
$$= -2\sqrt{1-x} - 2\int \left[\frac{1}{1+t^2} - \frac{1}{(1+t^2)^2}\right]\mathrm{d}t = -2\sqrt{1-x} - 2\left[\operatorname{arctg} t - \frac{1}{2}\operatorname{arctg} t - \frac{1}{2}\frac{t}{t^2+1}\right] + c$$
$$= \left[\begin{array}{c} t^2+1=\frac{x}{1-x}+1=\frac{x+1-x}{1-x}=\frac{1}{1-x} \\ \frac{t}{t^2+1}=(1-x)\sqrt{\frac{x}{1-x}}=\sqrt{(1-x)x}=\sqrt{x-x^2}\end{array}\right] = -2\sqrt{1-x} - \operatorname{arctg} t + \frac{t}{t^2+1} + c$$
$$= -2\sqrt{1-x} - \operatorname{arctg}\sqrt{\frac{x}{1-x}} + \sqrt{x-x^2} + c, \text{ for } x\in\langle 0\,;\,1).$$

$$\int f(x, \sqrt{ax^2+bx+c})\,\mathrm{d}x \quad \textbf{for } a\neq 0 \qquad \textbf{[Euler substitutions \dots\ ES]}$$

$\left[\text{1st ES } \sqrt{ax^2+bx+c}=t\pm x\sqrt{a} \text{ for } a>0\right]$ $\quad t=\sqrt{ax^2+bx+c}\mp x\sqrt{a},$

$ax^2+bx+c=t^2\pm 2tx\sqrt{a}+ax^2,\quad x=\frac{t^2-c}{b\mp 2t\sqrt{a}},\quad \mathrm{d}x=2\frac{\mp t^2\sqrt{a}+tb\mp c\sqrt{a}}{(b\mp 2t\sqrt{a})^2}\,\mathrm{d}t$

$\left[\text{2nd ES } \sqrt{ax^2+bx+c}=xt\pm\sqrt{c} \text{ for } c>0\right]$ $\quad t=\frac{\sqrt{ax^2+bx+c}\mp\sqrt{c}}{x},$

$ax^2+bx+c=x^2t^2\pm 2tx\sqrt{c}+c,\quad x=\frac{\pm 2t\sqrt{c}-b}{a-t^2}\ (x\neq 0),\quad \mathrm{d}x=2\frac{\pm t^2\sqrt{c}-tb\pm a\sqrt{c}}{(a-t^2)^2}\,\mathrm{d}t$

$\left[\text{3rd ES } t=\sqrt{a\frac{x-u}{x-v}} \text{ for } u,v \text{ real roots of the } ax^2+bx+c=0\right]$ $\quad x=\frac{vt^2-au}{t^2-a},\quad \mathrm{d}x=\frac{2ta(u-v)}{(t^2-a)^2}\,\mathrm{d}t$

$$\int \frac{\mathrm{d}x}{\sqrt{x^2+a^2}} = \left[\begin{array}{c} \boxed{\text{1st ES}}\quad \sqrt{x^2+a^2}=t-x,\ x^2+a^2=t^2-2tx+x^2,\ 2tx=t^2-a^2,\ x=\frac{t^2-a^2}{2t} \\ \sqrt{x^2+a^2}=t-\frac{t^2-a^2}{2t}=\frac{t^2+a^2}{2t},\ \mathrm{d}x=\frac{2t\cdot 2t-2(t^2-a^2)}{4t^2}\,\mathrm{d}t=\frac{2t^2+2a^2}{4t^2}\,\mathrm{d}t=\frac{t^2+a^2}{2t^2}\,\mathrm{d}t \\ \sqrt{a^2+x^2}>\sqrt{x^2}\geq x,\ \sqrt{a^2+x^2}-x>0,\ \ln\left|x+\sqrt{a^2+x^2}\right|=\ln\left(x+\sqrt{a^2+x^2}\right)\end{array}\right]$$
$$= \int \frac{2t}{t^2+a^2}\,\frac{t^2+a^2}{2t^2}\,\mathrm{d}t = \int \frac{\mathrm{d}t}{t} = \ln|t|+c = \ln\left|x+\sqrt{x^2+a^2}\right|+c$$
$$= \ln\left(x+\sqrt{x^2+a^2}\right)+c, \text{ for } a\in R,\ x\in R.$$

$$\int\frac{dx}{\sqrt{a^2-x^2}}=\left[\begin{array}{l}\sqrt{a^2-x^2}=xt+a,\ a^2-x^2=x^2t^2+2txa+a^2,\ x=\frac{-2ta}{t^2+1}, x\neq 0,\ t=\frac{\sqrt{a^2-x^2}-a}{x}\\ \boxed{\text{2nd ES}}\quad \sqrt{a^2-x^2}=\frac{-2t^2a}{t^2+1}+a=\frac{a(1-t^2)}{t^2+1},\ dx=\frac{-2a(t^2+1)+2ta\cdot 2t}{(t^2+1)^2}\,dt=\frac{2a(t^2-1)}{(t^2+1)^2}\,dt\end{array}\right]$$
$$=\int\frac{t^2+1}{a(1-t^2)}\frac{2a(t^2-1)}{(t^2+1)^2}\,dt=-2\int\frac{dt}{t^2+1}=-2\operatorname{arctg}t+c=-2\operatorname{arctg}\frac{\sqrt{a^2-x^2}-a}{x}+c,$$
for $a>0$, $x\in(-a\,;\,a)-\{0\}$.

---

$$\int\frac{dx}{\sqrt{x(1-x)}}=\left[\begin{array}{l}x(1-x)=x-x^2>0,\ x\in(0\,;\,1),\ t=\sqrt{\frac{x}{1-x}}>0,\ t^2=\frac{x}{1-x},\ t^2-t^2x=x,\ x=\frac{t^2}{t^2+1}\\ \boxed{\text{3rd ES}}\quad dx=\frac{2t(t^2+1)-t^2\cdot 2t}{(t^2+1)^2}\,dt=\frac{2t\,dt}{(t^2+1)^2},\ x(1-x)=\frac{t^2}{t^2+1}\left(1-\frac{t^2}{t^2+1}\right)=\frac{t^2}{(t^2+1)^2}\end{array}\right]$$
$$=\int\frac{t^2+1}{t}\frac{2t\,dt}{(t^2+1)^2}=\int\frac{2\,dt}{t^2+1}=2\operatorname{arctg}t+c=2\operatorname{arctg}\sqrt{\frac{x}{1-x}}+c,\ \text{for } x\in(0\,;\,1).$$

---

## $\int f(\sin x,\cos x)\,dx$ **[Universal trig substitution . . . UTS]**

$\left[\text{UTS}\ t=\operatorname{tg}\frac{x}{2},\ x=2\operatorname{arctg}t,\ x\in(-\pi+2k\pi\,;\,\pi+2k\pi), k\in Z,\ t\in R\right]$ $\quad dx=\frac{2\,dt}{t^2+1},$

$$\sin x=\left[2\sin\tfrac{x}{2}\cos\tfrac{x}{2}=\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}=\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}\cdot\frac{\cos^{-2}\frac{x}{2}}{\cos^{-2}\frac{x}{2}}=\frac{2\operatorname{tg}\frac{x}{2}}{\operatorname{tg}^2\frac{x}{2}+1}\right]=\frac{2t}{t^2+1}$$
$$\cos x=\left[\cos^2\tfrac{x}{2}-\sin^2\tfrac{x}{2}=\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}=\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}\cdot\frac{\cos^{-2}\frac{x}{2}}{\cos^{-2}\frac{x}{2}}=\frac{1-\operatorname{tg}^2\frac{x}{2}}{\operatorname{tg}^2\frac{x}{2}+1}\right]=\frac{1-t^2}{t^2+1}$$

---

$$\int\frac{dx}{\cos x}=\left[t=\operatorname{tg}\tfrac{x}{2},\ dx=\frac{2\,dt}{1+t^2},\ \cos x=\frac{1-t^2}{1+t^2}\right]=\int\frac{1+t^2}{1-t^2}\frac{2\,dt}{1+t^2}=\int\frac{2\,dt}{1-t^2}=-\int\frac{2\,dt}{t^2-1}$$
$$=-\frac{2}{2}\ln\left|\frac{t-1}{t+1}\right|+c=\ln\left|\frac{t+1}{t-1}\right|+c=\ln\left|\frac{\operatorname{tg}\frac{x}{2}+1}{\operatorname{tg}\frac{x}{2}-1}\right|+c,\ \text{for } x\in R-\left\{(2k+1)\tfrac{\pi}{2}\,;\,k\in Z\right\}.$$

---

$$\int\frac{dx}{1+\cos x}=\left[t=\operatorname{tg}\tfrac{x}{2},\ dx=\frac{2\,dt}{1+t^2},\ 1+\cos x=1+\frac{1-t^2}{1+t^2}=\frac{2}{1+t^2}\right]=\int\frac{1+t^2}{2}\frac{2\,dt}{1+t^2}=\int dt=t+c$$
$$=\operatorname{tg}\tfrac{x}{2}+c,\ \text{for } x\in R-\{\pi+2k\pi\,;\,k\in Z\}.$$

---

$$\int\frac{dx}{\sin x}=\left[t=\operatorname{tg}\tfrac{x}{2},\ dx=\frac{2\,dt}{1+t^2},\ \sin x=\frac{2t}{1+t^2}\right]=\int\frac{1+t^2}{2t}\frac{2\,dt}{1+t^2}=\int\frac{dt}{t}=\ln|t|+c$$
$$=\ln\left|\operatorname{tg}\tfrac{x}{2}\right|+c,\ \text{for } x\in R-\{k\pi\,;\,k\in Z\}.$$

---

$$\int\frac{dx}{1+\sin x}=\left[t=\operatorname{tg}\tfrac{x}{2},\ dx=\frac{2\,dt}{1+t^2},\ 1+\sin x=1+\frac{2t}{1+t^2}=\frac{1+2t+t^2}{1+t^2}\right]=\int\frac{1+t^2}{(1+t)^2}\frac{2\,dt}{1+t^2}=\int\frac{2\,dt}{(1+t)^2}$$
$$=\left[\begin{array}{c}1+t=u\\ dt=du\end{array}\right]=2\int\frac{du}{u^2}=2\int u^{-2}\,du=2\frac{u^{-1}}{-1}+c=c-\frac{2}{t+1}=c-\frac{2}{\operatorname{tg}\frac{x}{2}+1},$$
for $x\in R-\left\{\frac{\pi}{2}+k\pi\,;\,k\in Z\right\}$.

---

$\int f(\sin x, \cos x)\,\mathrm{d}x$ **[Universal hyperbolic substitution … UHS]**

$\left[\text{ UHS } t = \operatorname{tgh}\frac{x}{2},\ x = 2\operatorname{argtgh} t = \frac{1}{2}\ln\frac{1+t}{1-t},\ x\in R,\ t\in(-1\,;\,1)\ \right]\quad \mathrm{d}t = \frac{\mathrm{d}x}{2\cosh^2\frac{x}{2}},$

$$\mathrm{d}x = \left[\ 2\cosh^2\tfrac{x}{2}\,\mathrm{d}t = \frac{2\cosh^2\frac{x}{2}\,\mathrm{d}t}{\cosh^2\frac{x}{2}-\sinh^2\frac{x}{2}} = \frac{2\cosh^2\frac{x}{2}\,\mathrm{d}t}{\cosh^2\frac{x}{2}-\sinh^2\frac{x}{2}}\cdot\frac{\cos^{-2}\frac{x}{2}}{\cos^{-2}\frac{x}{2}} = \frac{2\,\mathrm{d}t}{1-\operatorname{tg}^2\frac{x}{2}}\ \right] = \frac{2\,\mathrm{d}t}{1-t^2}$$

$$\sinh x = \left[\ 2\sinh\tfrac{x}{2}\cosh\tfrac{x}{2} = \frac{2\sinh\frac{x}{2}\cosh\frac{x}{2}}{\cosh^2\frac{x}{2}-\sinh^2\frac{x}{2}} = \frac{2\sinh\frac{x}{2}\cosh\frac{x}{2}}{\cosh^2\frac{x}{2}-\sinh^2\frac{x}{2}}\cdot\frac{\cosh^{-2}\frac{x}{2}}{\cosh^{-2}\frac{x}{2}} = \frac{2\operatorname{tgh}\frac{x}{2}}{1-\operatorname{tgh}^2\frac{x}{2}}\ \right] = \frac{2t}{1-t^2}$$

$$\cosh x = \left[\ \cosh^2\tfrac{x}{2}+\sinh^2\tfrac{x}{2} = \frac{\cosh^2\frac{x}{2}+\sinh^2\frac{x}{2}}{\cosh^2\frac{x}{2}-\sinh^2\frac{x}{2}} = \frac{\cosh^2\frac{x}{2}+\sinh^2\frac{x}{2}}{\cosh^2\frac{x}{2}-\sinh^2\frac{x}{2}}\cdot\frac{\cosh^{-2}\frac{x}{2}}{\cosh^{-2}\frac{x}{2}} = \frac{1+\operatorname{tg}^2\frac{x}{2}}{1-\operatorname{tg}^2\frac{x}{2}}\ \right] = \frac{1+t^2}{1-t^2}$$

---

$$\int\frac{\mathrm{d}x}{\sinh x} = \left[\begin{array}{c} t=\operatorname{tgh}\frac{x}{2},\ \mathrm{d}x=\frac{2\,\mathrm{d}t}{1-t^2},\ x\in R-\{0\} \\ \sinh x=\frac{2t}{1-t^2},\ t\in(-1\,;\,1)-\{0\}\end{array}\right] = \int\frac{1-t^2}{2t}\,\frac{2\,\mathrm{d}t}{1-t^2} = \int\frac{\mathrm{d}t}{t} = \ln|t|+c$$
$$= \ln\left|\operatorname{tgh}\tfrac{x}{2}\right| + c, \text{ for } x\in R-\{0\}.$$

---

$$\int\frac{\mathrm{d}x}{1+\sinh x} = \left[\begin{array}{c} t=\operatorname{tgh}\frac{x}{2},\ \sinh x=\frac{2t}{1-t^2} \\ \mathrm{d}x=\frac{2\,\mathrm{d}t}{1-t^2},\ x\in R,\ t\in(-1\,;\,1)\end{array}\right] = \int\frac{\frac{2\,\mathrm{d}t}{1-t^2}}{1+\frac{2t}{1-t^2}} = \int\frac{\frac{2\,\mathrm{d}t}{1-t^2}}{\frac{1-t^2+2t}{1-t^2}} = -\int\frac{2\,\mathrm{d}t}{t^2-2t-1}$$
$$= -\int\frac{2\,\mathrm{d}t}{(t-1)^2-2} = -\frac{2}{2\sqrt{2}}\ln\left|\frac{t-1-\sqrt{2}}{t-1+\sqrt{2}}\right| + c = -\frac{\sqrt{2}}{2}\ln\left|\frac{\operatorname{tgh}\frac{x}{2}-1-\sqrt{2}}{\operatorname{tgh}\frac{x}{2}-1+\sqrt{2}}\right| + c,$$
for $x\in R$, $\sinh x\neq 0$.

---

$$\int\frac{\mathrm{d}x}{\cosh x} = \left[\begin{array}{c} t=\operatorname{tgh}\frac{x}{2},\ \mathrm{d}x=\frac{2\,\mathrm{d}t}{1-t^2},\ x\in R \\ \cosh x=\frac{1+t^2}{1-t^2},\ t\in(-1\,;\,1)\end{array}\right] = \int\frac{1-t^2}{1+t^2}\,\frac{2\,\mathrm{d}t}{1-t^2} = \int\frac{2\,\mathrm{d}t}{1+t^2} = 2\operatorname{arctg} t + c$$
$$= 2\operatorname{arctg}\operatorname{tgh}\tfrac{x}{2} + c, \text{ for } x\in R.$$

---

$$\int\frac{\mathrm{d}x}{1+\cosh x} = \left[\begin{array}{c} t=\operatorname{tgh}\frac{x}{2},\ \cosh x=\frac{1+t^2}{1-t^2} \\ \mathrm{d}x=\frac{2\,\mathrm{d}t}{1-t^2},\ x\in R,\ t\in(-1\,;\,1)\end{array}\right] = \int\frac{\frac{2\,\mathrm{d}t}{1-t^2}}{1+\frac{1+t^2}{1-t^2}} = \int\frac{\frac{2\,\mathrm{d}t}{1-t^2}}{\frac{2}{1-t^2}} = \int\mathrm{d}t = t+c$$
$$= \operatorname{tgh}\tfrac{x}{2} + c, \text{ for } x\in R.$$

---

$$\int\frac{\mathrm{d}x}{1+\cosh x} = \left[\ \frac{1}{\cosh x+1}\,\frac{\cosh x-1}{\cosh x-1} = \frac{\cosh x-1}{\cosh^2 x-1} = \frac{\cosh x-1}{\sinh^2 x}\ \right] = \int\frac{\cosh x\,\mathrm{d}x}{\sinh^2 x} - \int\frac{\mathrm{d}x}{\sinh^2 x}$$
$$= \left[\begin{array}{c|c} t=\sinh x & x<0\Rightarrow t<0 \\ \mathrm{d}t=\cosh x\,\mathrm{d}x & x>0\Rightarrow t>0\end{array}\right] = \int\frac{\mathrm{d}t}{t^2} - \operatorname{cotgh} x = \frac{1}{-t} - \operatorname{cotgh} x + c$$
$$= -\frac{1}{t} - \operatorname{cotgh} x + c = -\frac{1}{\sinh x} - \frac{\cosh x}{\sinh x} + c = -\frac{1+\cosh x}{\sinh x} + c, \text{ for } x\in R-\{0\}.$$

---

$$\int\frac{\mathrm{d}x}{1+\cosh x} = \int\frac{\mathrm{d}x}{2\cosh^2\frac{x}{2}} = \left[\begin{array}{c} t=\frac{x}{2},\quad x\in R \\ \mathrm{d}t=\frac{\mathrm{d}x}{2},\ t\in R\end{array}\right] = \int\frac{\mathrm{d}t}{\cosh^2 t} = \operatorname{tgh} t + c = \operatorname{tgh}\tfrac{x}{2} + c, \text{ for } x\in R.$$

---

$$\int\frac{\mathrm{d}x}{1+\cosh x} = \int\frac{\mathrm{d}x}{1+\frac{\mathrm{e}^x+\mathrm{e}^{-x}}{2}} = \int\frac{2\,\mathrm{d}x}{2+\mathrm{e}^x+\mathrm{e}^{-x}} = \left[\begin{array}{c} t=\mathrm{e}^x,\ x=\ln t,\ \mathrm{d}x=\frac{\mathrm{d}t}{t} \\ \mathrm{e}^{-x}=\frac{1}{t},\ x\in R,\ t\in(0\,;\,\infty)\end{array}\right] = \int\frac{2\,\mathrm{d}t}{t(2+t+\frac{1}{t})}$$
$$= \int\frac{2\,\mathrm{d}t}{t^2+2t+1} = \int\frac{2\,\mathrm{d}t}{(t+1)^2} = \int 2(t+1)^{-2}\,\mathrm{d}t = 2\frac{(t+1)^{-1}}{-1} + c = -\frac{2}{\mathrm{e}^x+1} + c, \text{ for } x\in R.$$

---

beerb@frcatel.fri.uniza.sk 

# 4 Integration by parts (per partes)

**Theorem.**

Let $f, g$ be functions differentiable on an interval $I$.
Then $fg'$ is integrable on $I$ if and only if $f'g$ is integrable on $I$, and

$$\forall x \in I\colon \int f'(x)g(x)\,\mathrm{d}x = f(x)g(x) - \int f(x)g'(x)\,\mathrm{d}x.$$

---

$$\int (x+1)\,\mathrm{e}^x\,\mathrm{d}x = \left[\begin{array}{l|l} u = x+1 & u' = 1 \\ v' = \mathrm{e}^x & v = \mathrm{e}^x \end{array}\right] = (x+1)\,\mathrm{e}^x - \int \mathrm{e}^x\,\mathrm{d}x = (x+1)\,\mathrm{e}^x - \mathrm{e}^x + c$$
$$= x\,\mathrm{e}^x + c, \text{ for } x \in R.$$

---

$$\int \operatorname{arctg} x\,\mathrm{d}x = \left[\begin{array}{l|l} u' = 1 & u = x \\ v = \operatorname{arctg} x & v' = \frac{1}{1+x^2} \end{array}\right] = x \operatorname{arctg} x - \int \frac{x}{1+x^2}\,\mathrm{d}x = x \operatorname{arctg} x - \frac{1}{2}\int \frac{2x}{1+x^2}\,\mathrm{d}x$$
$$= x \operatorname{arctg} x - \frac{1}{2}\ln\left|1+x^2\right| + c = x \operatorname{arctg} x - \frac{1}{2}\ln\left(1+x^2\right) + c$$
$$= x \operatorname{arctg} x - \ln\sqrt{1+x^2} + c, \text{ for } x \in R.$$

---

$$\int x \operatorname{arctg} x\,\mathrm{d}x = \left[\begin{array}{l|l} u' = x & u = \frac{x^2}{2} \\ v = \operatorname{arctg} x & v' = \frac{1}{1+x^2} \end{array}\right] = \frac{x^2}{2}\operatorname{arctg} x - \frac{1}{2}\int \frac{x^2\,\mathrm{d}x}{1+x^2}$$
$$= \frac{x^2}{2}\operatorname{arctg} x - \frac{1}{2}\int \frac{(1+x^2-1)\,\mathrm{d}x}{1+x^2} = \frac{x^2}{2}\operatorname{arctg} x - \frac{1}{2}\int \mathrm{d}x + \frac{1}{2}\int \frac{\mathrm{d}x}{1+x^2}$$
$$= \frac{x^2}{2}\operatorname{arctg} x - \frac{x}{2} + \frac{1}{2}\operatorname{arctg} x + c = \left(\frac{x^2}{2} + \frac{1}{2}\right)\operatorname{arctg} x - \frac{x}{2} + c, \text{ for } x \in R.$$

---

$$\int \ln x\,\mathrm{d}x = \left[\begin{array}{l|l} u = \ln x & u' = \frac{1}{x} \\ v' = 1 & v = x \end{array}\right] = x\ln x - \int \frac{x}{x}\,\mathrm{d}x = x\ln x - \int \mathrm{d}x = x\ln x - x + c,$$
for $x > 0$.

---

$$\int x\ln x\,\mathrm{d}x = \left[\begin{array}{l|l} u = \ln x & u' = \frac{1}{x} \\ v' = x & v = \frac{x^2}{2} \end{array}\right] = \frac{x^2}{2}\ln x - \int \frac{x^2}{2}\frac{1}{x}\,\mathrm{d}x = \frac{x^2}{2}\ln x - \int \frac{x}{2}\,\mathrm{d}x$$
$$= \frac{x^2}{2}\ln x - \frac{1}{2}\cdot\frac{x^2}{2}\,\mathrm{d}x = \frac{x^2}{2}\ln x - \frac{x^2}{4} + c, \text{ for } x > 0.$$

---

$$\int x^n \ln x\,\mathrm{d}x = \left[\begin{array}{l|l} u = \ln x & u' = \frac{1}{x} \\ v' = x^n & v = \frac{x^{n+1}}{n+1} \end{array}\right] = \frac{x^{n+1}}{n+1}\ln x - \int \frac{x^{n+1}}{n+1}\frac{1}{x}\,\mathrm{d}x = \frac{x^{n+1}}{n+1}\ln x - \frac{1}{n+1}\int x^n\,\mathrm{d}x$$
$$= \frac{x^{n+1}}{n+1}\ln x - \frac{1}{n+1}\cdot\frac{x^{n+1}}{n+1} = \frac{x^{n+1}}{n+1}\ln x - \frac{x^{n+1}}{(n+1)^2} + c, \text{ for } x > 0,\ n \in N.$$

---

$$\int \frac{\ln x}{\sqrt{x}}\,\mathrm{d}x = \left[\begin{array}{l|l} u = \ln x & u' = \frac{1}{x} \\ v' = x^{-\frac{1}{2}} & v = 2x^{\frac{1}{2}} \end{array}\right] = 2\sqrt{x}\ln x - 2\int \frac{x^{\frac{1}{2}}}{x}\,\mathrm{d}x = 2\sqrt{x}\ln x - 2\int x^{-\frac{1}{2}}\,\mathrm{d}x$$
$$= 2\sqrt{x}\ln x - 2\cdot 2x^{\frac{1}{2}} + c = 2\sqrt{x}\ln x - 4\sqrt{x} + c, \text{ for } x > 0.$$

---

$$\int x\ln^2 x\,dx = \left[\begin{array}{l|l} u=\ln^2 x & u'=\frac{2\ln x}{x} \\ v'=x & v=\frac{x^2}{2}\end{array}\right] = \frac{x^2}{2}\ln^2 x - \int \frac{x^2}{2}\frac{2\ln x}{x}\,dx = \frac{x^2}{2}\ln^2 x - \int x\ln x\,dx$$
$$= \left[\begin{array}{l|l} u=\ln x & u'=\frac{1}{x} \\ v'=x & v=\frac{x^2}{2}\end{array}\right] = \frac{x^2}{2}\ln^2 x - \left[\frac{x^2}{2}\ln x - \int \frac{x^2}{2}\frac{1}{x}\,dx\right] = \frac{x^2}{2}\ln^2 x - \frac{x^2}{2}\ln x + \int \frac{x\,dx}{2}$$
$$= \frac{x^2}{2}\ln^2 x - \frac{x^2}{2}\ln x + \frac{x^2}{4} + c, \text{ for } x>0.$$

---

$$I_n = \int \sin^n x\,dx = \int \sin x \sin^{n-1} x\,dx = \left[\begin{array}{l|l} u=\sin^{n-1} x & u'=(n-1)\sin^{n-2} x\cos x \\ v'=\sin x & v=-\cos x\end{array}\right]$$
$$= -\cos x\sin^{n-1} x + (n-1)\int \sin^{n-2} x\cos^2 x\,dx$$
$$= -\cos x\sin^{n-1} x + (n-1)\int \sin^{n-2} x(1-\sin^2 x)\,dx$$
$$= -\cos x\sin^{n-1} x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx$$
$$= \left[\begin{array}{l} \boxed{\text{Equation}} \quad I_n = -\cos x\sin^{n-1} x + (n-1)I_{n-2} - (n-1)I_n \\ \quad \Rightarrow\ I_n + (n-1)I_n = -\cos x\sin^{n-1} x + (n-1)I_{n-2} \ \Rightarrow\ I_n = -\frac{\cos x\sin^{n-1} x}{n} + \frac{n-1}{n}I_{n-2}\end{array}\right]$$
$$= -\frac{\cos x\sin^{n-1} x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx, \text{ for } x\in R,\ n=3,4,5,\ldots.$$

---

$$I_1 = \int \sin x\,dx \quad = -\cos x + c, \text{ for } x\in R.$$

$$I_2 = \int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx = \frac{x}{2} - \frac{\sin 2x}{4} + c, \text{ for } x\in R.$$

---

$$I_n = \int \cos^n x\,dx = \int \cos x \cos^{n-1} x\,dx = \left[\begin{array}{l|l} u=\cos^{n-1} x & u'=-(n-1)\cos^{n-2} x\sin x \\ v'=\cos x & v=\sin x\end{array}\right]$$
$$= \sin x\cos^{n-1} x + (n-1)\int \cos^{n-2} x\sin^2 x\,dx$$
$$= \sin x\cos^{n-1} x + (n-1)\int \cos^{n-2} x(1-\cos^2 x)\,dx =$$
$$= \sin x\cos^{n-1} x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx$$
$$= \left[\begin{array}{l} \boxed{\text{Equation}} \quad I_n = \sin x\cos^{n-1} x + (n-1)I_{n-2} - (n-1)I_n \\ \quad \Rightarrow\ I_n + (n-1)I_n = \sin x\cos^{n-1} x + (n-1)I_{n-2} \ \Rightarrow\ I_n = \frac{\sin x\cos^{n-1} x}{n} + \frac{n-1}{n}I_{n-2}\end{array}\right]$$
$$= \frac{\sin x\cos^{n-1} x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx, \text{ for } x\in R,\ n=3,4,5,\ldots.$$

---

$$I_1 = \int \cos x\,dx \quad = \sin x + c, \text{ for } x\in R.$$

$$I_2 = \int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx = \frac{x}{2} + \frac{\sin 2x}{4} + c, \text{ for } x\in R.$$

---

# 5 Partial fractions

$$\int \frac{dx}{(x-a)^n} = \left[ \begin{matrix} x-a=t \\ dx=du \end{matrix} \right] = \int \frac{dt}{t^n}$$
$$= \begin{cases} \int \frac{dt}{t} = \ln|t| + c = \ln|x-a| + c, \text{ for } a\in R,\ x\in R-\{a\},\ n=1, \\ \int t^{-n}\,dt = \frac{t^{1-n}}{1-n} + c = \frac{(x-a)^{1-n}}{1-n} + c, \text{ for } a\in R,\ x\in R-\{a\},\ n=2,3,4,\dots. \end{cases}$$

$$\int \frac{dx}{(x+1)^n}\,dx = \left[ \begin{matrix} x+1=t \\ dx=dt \end{matrix} \right] = \int t^{-n}\,dt = \frac{t^{1-n}}{1-n} + c = \frac{(x+1)^{1-n}}{1-n} + c = \frac{1}{(1-n)(x+1)^{n-1}} + c,$$
for $n=2,3,4,\dots,\ x\in R-\{-1\}$.

$$\int \frac{dx}{(x-1)^n}\,dx = \left[ \begin{matrix} x-1=t \\ dx=dt \end{matrix} \right] = \int t^{-n}\,dt = \frac{t^{1-n}}{1-n} + c = \frac{(x-1)^{1-n}}{1-n} + c = \frac{1}{(1-n)(x-1)^{n-1}} + c,$$
for $n=2,3,4,\dots,\ x\in R-\{1\}$.

$$\int \frac{dx}{x^2-a^2} = \int \frac{1}{2a}\left[\frac{1}{x-a} - \frac{1}{x+a}\right] dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c, \text{ for } a>0,\ x\in R-\{\pm a\}.$$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + c = -\frac{1}{a}\operatorname{arccotg}\frac{x}{a} + c, \text{ for } a>0,\ x\in R.$$

$$\int \frac{dx}{x^2+b} = \begin{cases} \frac{1}{\sqrt{b}}\operatorname{arctg}\frac{x}{\sqrt{b}} + c = -\frac{1}{\sqrt{b}}\operatorname{arccotg}\frac{x}{\sqrt{b}} + c, \text{ for } b>0,\ x\in R, \\ \frac{1}{2\sqrt{-b}}\ln\left|\frac{x-\sqrt{-b}}{x+\sqrt{-b}}\right| + c, \text{ for } b<0,\ x\in R-\{\pm\sqrt{-b}\}. \end{cases}$$

$$\int \frac{dx}{x^2+ax+b} = \left[ \begin{matrix} x^2+ax+b = (x+\frac{a}{2})^2 + b - \frac{a^2}{4} \\ t=x+\frac{a}{2},\ dt=dx,\ \alpha^2=\pm(b-\frac{a^2}{4}),\ \alpha\geq 0 \end{matrix} \right] = \int \frac{dt}{t^2\pm\alpha^2}$$
$$= \begin{cases} \int \frac{dt}{t^2-\alpha^2} = \frac{1}{2\alpha}\ln\left|\frac{t-\alpha}{t+\alpha}\right| + c = \frac{1}{2\alpha}\ln\left|\frac{x+\frac{a}{2}-\sqrt{\frac{a^2}{4}-b}}{x+\frac{a}{2}+\sqrt{\frac{a^2}{4}-b}}\right| + c, \text{ for } 4b-a^2<0, \\ \int \frac{dt}{t^2} = \frac{t^{-2+1}}{-2+1} + c = -\frac{1}{t} + c = -\frac{1}{x+\frac{a}{2}} + c, \text{ for } 4b-a^2=0, \\ \int \frac{dt}{t^2+\alpha^2} = \frac{1}{\alpha}\operatorname{arctg}\frac{t}{\alpha} + c = \frac{1}{\sqrt{b-\frac{a^2}{4}}}\operatorname{arctg}\frac{x+\frac{a}{2}}{\sqrt{b-\frac{a^2}{4}}} + c, \text{ for } 4b-a^2>0, \end{cases}$$
for $a,b\in R,\ x\in R,\ x^2+ax+b\neq 0$.

$$\begin{aligned}
I_n &= \int \frac{dx}{(x^2 \pm a^2)^n} = \frac{1}{\pm a^2} \int \frac{\pm a^2\, dx}{(x^2 \pm a^2)^n} = \frac{1}{\pm a^2} \int \frac{x^2 \pm a^2 - x^2}{(x^2 \pm a^2)^n}\, dx \\
&= \frac{1}{\pm a^2} \int \frac{x^2 \pm a^2}{(x^2 \pm a^2)^n}\, dx - \frac{1}{\pm a^2} \int \frac{x^2\, dx}{(x^2 \pm a^2)^n} = \frac{1}{\pm a^2} \int \frac{dx}{(x^2 \pm a^2)^{n-1}} - \frac{1}{\pm 2a^2} \int \frac{x \cdot 2x\, dt}{(x^2 \pm a^2)^n} \\
&= \left[ \begin{array}{l|l} u = x & u' = 1 \\ v' = \frac{2x}{(x^2 \pm a^2)^n} & v = \frac{(x^2 \pm a^2)^{1-n}}{1-n} = \frac{1}{(1-n)(x^2 \pm a^2)^{n-1}} \end{array} \right] \\
&= \frac{1}{\pm a^2} \int \frac{dx}{(x^2 \pm a^2)^{n-1}} - \frac{1}{\pm 2a^2} \left[ \frac{x}{(1-n)(x^2 \pm a^2)^{n-1}} - \frac{1}{1-n} \int \frac{dx}{(x^2 \pm a^2)^{n-1}} \right] \\
&= \frac{1}{\pm a^2} \int \frac{dx}{(x^2 \pm a^2)^{n-1}} - \frac{x}{\pm 2a^2(1-n)(x^2 \pm a^2)^{n-1}} + \frac{1}{\pm 2a^2} \frac{1}{1-n} \int \frac{dx}{(x^2 \pm a^2)^{n-1}} \\
&= \frac{1}{\pm 2a^2} \left(2 + \frac{1}{1-n}\right) \int \frac{dx}{(x^2 \pm a^2)^{n-1}} - \frac{x}{\pm 2a^2(1-n)(x^2 \pm a^2)^{n-1}} \\
&= \frac{3-2n}{\pm 2a^2(1-n)} \int \frac{dx}{(x^2 \pm a^2)^{n-1}} - \frac{x}{\pm 2a^2(1-n)(x^2 \pm a^2)^{n-1}} \\
&= \frac{3-2n}{\pm 2a^2(1-n)} I_{n-1} - \frac{x}{\pm 2a^2(1-n)(x^2 \pm a^2)^{n-1}}, \text{ for } a > 0,\ n = 2, 3, 4, \ldots,\ x \in R,\ x^2 \pm a^2 \neq 0.
\end{aligned}$$

---

$$\int \frac{dx}{(x^2 - a^2)^n} = \frac{3-2n}{2a^2(n-1)} \int \frac{dx}{(x^2 - a^2)^{n-1}} - \frac{x}{2a^2(n-1)(x^2 - a^2)^{n-1}},$$

for $a > 0$, $n = 2, 3, 4, \ldots$, $x \in R - \{\pm a\}$.

$$\int \frac{dx}{(x^2 + a^2)^n} = \frac{3-2n}{2a^2(1-n)} \int \frac{dx}{(x^2 + a^2)^{n-1}} + \frac{x}{2a^2(n-1)(x^2 + a^2)^{n-1}}, \text{ for } a > 0,\ n = 2, 3, 4, \ldots,\ x \in R.$$

---

$$\begin{aligned}
I_n &= \int \frac{dx}{(x^2 + ax + b)^n} = \left[ \begin{array}{c} x^2 + ax + b = (x + \frac{a}{2})^2 + b - \frac{a^2}{4} \\ t = x + \frac{a}{2},\ dt = dx,\ \alpha^2 = \pm(b - \frac{a^2}{4}),\ \alpha \geq 0 \end{array} \right] = \int \frac{dt}{t^2 \pm \alpha^2} \\
&= \frac{3-2n}{\pm 2\alpha^2(1-n)} \int \frac{dt}{(t^2 \pm \alpha^2)^{n-1}} - \frac{t}{\pm 2\alpha^2(1-n)(t^2 \pm \alpha^2)^{n-1}} \\
&= \frac{3-2n}{2(b - \frac{a^2}{4})(1-n)} \int \frac{dx}{(x^2 + ax + b)^{n-1}} - \frac{x + \frac{a}{2}}{2(b - \frac{a^2}{4})(1-n)(x^2 + ax + b)^{n-1}} \\
&= \frac{3-2n}{2(b - \frac{a^2}{4})(1-n)} I_{n-1} - \frac{x + \frac{a}{2}}{2(b - \frac{a^2}{4})(1-n)(x^2 + ax + b)^{n-1}},
\end{aligned}$$

for $a, b \in R$, $n = 2, 3, 4, \ldots$, $x \in R$, $x \neq \frac{-a \pm \sqrt{a^2 - 4b}}{2}$.

---

$$\begin{aligned}
\int \frac{2x + a}{(x^2 + ax + b)^n}\, dx &= \left[ \begin{array}{c} x^2 + ax + b = u \\ (2x + a)\, dx = du \end{array} \right] = \int \frac{du}{u^n} \\
&= \begin{cases} \ln|u| + c = \ln\left|x^2 + ax + b\right| + c, & \text{for } n = 1, \\ \frac{u^{1-n}}{1-n} + c = \frac{1}{(1-n)(x^2 + ax + b)^{n-1}} + c, & \text{for } n = 2, 3, 4, \ldots, a, b \in R, x \in R, x \neq \frac{-a \pm \sqrt{a^2 - 4b}}{2}. \end{cases}
\end{aligned}$$

---

$$\int \frac{x + A}{(x^2 + ax + b)^n}\, dx = \frac{1}{2} \int \frac{2x + a + 2A - a}{(x^2 + ax + b)^n}\, dx = \frac{1}{2} \int \frac{2x + a}{(x^2 + ax + b)^n}\, dx + \frac{2A - a}{2} \int \frac{dx}{(x^2 + ax + b)^n}$$

for $n = 2, 3, 4 \ldots$, $a, b \in R$, $x \in R$, $x \neq \frac{-a \pm \sqrt{a^2 - 4b}}{2}$.

---

$$I_n=\int\frac{\mathrm{d}x}{(x^2-a^2)^n}=\frac{3-2n}{2a^2(n-1)}I_{n-1}-\frac{x}{2a^2(n-1)(x^2-a^2)^{n-1}},$$
for $a>0$, $n=2,3,4,\dots$, $x\in R-\{\pm a\}$.

$$\int\frac{\mathrm{d}x}{(x^2-a^2)^2}=\frac{3-2\cdot 2}{2a^2(2-1)}\int\frac{\mathrm{d}x}{x^2-a^2}-\frac{x}{2a^2(2-1)(x^2-a^2)}=\frac{-1}{2a^2}\int\frac{\mathrm{d}x}{x^2-a^2}-\frac{x}{2a^2(x^2-a^2)}$$
$$=\frac{-1}{2a^2}\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|-\frac{x}{2a^2(x^2-a^2)}+c=\frac{-1}{4a^3}\ln\left|\frac{x-a}{x+a}\right|-\frac{x}{2a^2(x^2-a^2)}+c,$$
for $a>0$, $x\in R-\{\pm a\}$.

$$\int\frac{\mathrm{d}x}{(x^2-a^2)^3}=\frac{3-2\cdot 3}{2a^2(3-1)}\int\frac{\mathrm{d}x}{(x^2-a^2)^2}-\frac{x}{2a^2(3-1)(x^2-a^2)^2}=\frac{-3}{4a^2}\int\frac{\mathrm{d}x}{(x^2-a^2)^2}-\frac{x}{4a^2(x^2-a^2)^2}$$
$$=\frac{-3}{4a^2}\left[\frac{-1}{4a^3}\ln\left|\frac{x-a}{x+a}\right|-\frac{x}{2a^2(x^2-a^2)}\right]-\frac{x}{4a^2(x^2-a^2)^2}$$
$$=\frac{3}{16a^5}\ln\left|\frac{x-a}{x+a}\right|+\frac{3x}{8a^4(x^2-a^2)}-\frac{x}{4a^2(x^2-a^2)^2}+c,$$ for $a>0$, $x\in R-\{\pm a\}$.

$$\int\frac{\mathrm{d}x}{(x^2-a^2)^4}=\frac{3-2\cdot 4}{2a^2(4-1)}\int\frac{\mathrm{d}x}{(x^2-a^2)^3}-\frac{x}{2a^2(4-1)(x^2-a^2)^3}=\frac{-5}{6a^2}\int\frac{\mathrm{d}x}{(x^2-a^2)^3}-\frac{x}{6a^2(x^2-a^2)^3}$$
$$=\frac{-5}{6a^2}\left[\frac{3}{16a^5}\ln\left|\frac{x-a}{x+a}\right|+\frac{3x}{8a^4(x^2-a^2)}-\frac{x}{4a^2(x^2-a^2)^2}\right]-\frac{x}{6a^2(x^2-a^2)^3}$$
$$=\frac{-5}{32a^7}\ln\left|\frac{x-a}{x+a}\right|-\frac{5x}{16a^6(x^2-a^2)}+\frac{5x}{24a^4(x^2-a^2)^2}-\frac{x}{6a^2(x^2-a^2)^3},$$ for $a>0$, $x\in R-\{\pm a\}$.

$$I_n=\int\frac{\mathrm{d}x}{(x^2+a^2)^n}=\frac{3-2n}{2a^2(1-n)}I_{n-1}+\frac{x}{2a^2(n-1)(x^2+a^2)^{n-1}},$$ for $a>0$, $n=2,3,4,\dots$, $x\in R$.

$$\int\frac{\mathrm{d}x}{(x^2+a^2)^2}=\frac{3-2\cdot 2}{2a^2(1-2)}\int\frac{\mathrm{d}x}{x^2+a^2}+\frac{x}{2a^2(2-1)(x^2+a^2)}=\frac{1}{2a^2}\int\frac{\mathrm{d}x}{x^2+a^2}+\frac{x}{2a^2(x^2+a^2)}$$
$$=\frac{1}{2a^2}\frac{1}{a}\operatorname{arctg}\frac{x}{a}+\frac{x}{2a^2(x^2+a^2)}+c=\frac{1}{2a^3}\operatorname{arctg}\frac{x}{a}+\frac{x}{2a^2(x^2+a^2)}+c,$$ for $a>0$, $x\in R$.

$$\int\frac{\mathrm{d}x}{(x^2+a^2)^3}=\frac{3-2\cdot 3}{2a^2(1-3)}\int\frac{\mathrm{d}x}{(x^2+a^2)^2}+\frac{x}{2a^2(3-1)(x^2+a^2)^2}=\frac{3}{4a^2}\int\frac{\mathrm{d}x}{(x^2+a^2)^2}+\frac{x}{4a^2(x^2+a^2)^2}$$
$$=\frac{3}{4a^2}\left[\frac{1}{2a^3}\operatorname{arctg}\frac{x}{a}+\frac{x}{2a^2(x^2+a^2)}\right]+\frac{x}{4a^2(x^2+a^2)^2}+c$$
$$=\frac{3}{8a^5}\operatorname{arctg}\frac{x}{a}+\frac{3x}{8a^4(x^2+a^2)}+\frac{x}{4a^2(x^2+a^2)^2}+c,$$ for $a>0$, $x\in R$.

$$\int\frac{\mathrm{d}x}{(x^2+a^2)^4}=\frac{3-2\cdot 4}{2a^2(1-4)}\int\frac{\mathrm{d}x}{(x^2+a^2)^3}+\frac{x}{2a^2(4-1)(x^2+a^2)^3}=\frac{5}{6a^2}\int\frac{\mathrm{d}x}{(x^2+a^2)^3}+\frac{x}{6a^2(x^2+a^2)^3}$$
$$=\frac{5}{6a^2}\left[\frac{3}{8a^5}\operatorname{arctg}\frac{x}{a}+\frac{3x}{8a^4(x^2+a^2)}+\frac{x}{4a^2(x^2+a^2)^2}\right]+\frac{x}{6a^2(x^2+a^2)^3}+c$$
$$=\frac{5}{16a^7}\operatorname{arctg}\frac{x}{a}+\frac{5x}{16a^6(x^2+a^2)}+\frac{5x}{24a^4(x^2+a^2)^2}+\frac{x}{6a^2(x^2+a^2)^3}+c,$$ for $a>0$, $x\in R$.

$$I_n=\int\frac{dx}{(x^2+1)^n}=\int\frac{x^2+1-x^2}{(x^2+1)^n}\,dx=\int\frac{x^2+1}{(x^2+1)^n}\,dx-\int\frac{x^2\,dx}{(x^2+1)^n}=\int\frac{dx}{(x^2+1)^{n-1}}-\frac{1}{2}\int\frac{x\cdot 2x\,dt}{(x^2+1)^n}$$
$$=\left[\begin{array}{l|l} u=x & u'=1 \\ v'=\frac{2x}{(x^2+1)^n} & v=\frac{(x^2+1)^{1-n}}{1-n}=\frac{1}{(1-n)(x^2+1)^{n-1}}\end{array}\right]$$
$$=\int\frac{dx}{(x^2+1)^{n-1}}-\frac{1}{2}\left[\frac{x}{(1-n)(x^2+1)^{n-1}}-\frac{1}{1-n}\int\frac{dx}{(x^2+1)^{n-1}}\right]$$
$$=\int\frac{dx}{(x^2+1)^{n-1}}-\frac{x}{2(1-n)(x^2+1)^{n-1}}+\frac{1}{2(1-n)}\int\frac{dx}{(x^2+1)^{n-1}}$$
$$=\left(1+\frac{1}{2(1-n)}\right)\int\frac{dx}{(x^2+1)^{n-1}}-\frac{x}{2(1-n)(x^2+1)^{n-1}}=\frac{3-2n}{2(1-n)}\int\frac{dx}{(x^2+1)^{n-1}}-\frac{x}{2(1-n)(x^2+1)^{n-1}}$$
$$=\frac{3-2n}{2(1-n)}I_{n-1}-\frac{x}{2(1-n)(x^2+1)^{n-1}},\ \text{for } n=2,3,4,\ldots,\ x\in R.$$

---

$$\int\frac{dx}{(x^2+1)^2}=\int\frac{x^2+1-x^2}{(x^2+1)^2}\,dx=\int\frac{x^2+1}{(x^2+1)^2}\,dx-\int\frac{x^2\,dx}{(x^2+1)^2}=\int\frac{dx}{x^2+1}-\frac{1}{2}\int\frac{x\cdot 2x\,dt}{(x^2+1)^2}$$
$$=\left[\begin{array}{l|l} u=x & u'=1 \\ v'=\frac{2x}{(x^2+1)^2} & v=\frac{(x^2+1)^{1-2}}{1-2}=-\frac{1}{x^2+1}\end{array}\right]=\int\frac{dx}{x^2+1}-\frac{1}{2}\left[-\frac{x}{x^2+1}+\int\frac{dx}{x^2+1}\right]$$
$$=\frac{1}{2}\int\frac{dx}{x^2+1}+\frac{1}{2}\,\frac{x}{x^2+1}=\frac{1}{2}\operatorname{arctg}x+\frac{x}{2(x^2+1)}+c,\ \text{for } x\in R.$$

---

$$\int\frac{x\,dx}{(x^2+1)^n}=\left[\begin{array}{c} x^2+1=t \\ 2x\,dx=dt\end{array}\right]=\frac{1}{2}\int\frac{dt}{t^n}$$
$$=\begin{cases}\frac{1}{2}\,|t|+c=\frac{1}{2}\ln\left|x^2+1\right|+c=\frac{1}{2}\ln\left(x^2+1\right)+c, & \text{for } n=1,\ x\in R,\\ \frac{1}{2}\,\frac{t^{-n+1}}{-n+1}+c=\frac{1}{2(1-n)t^{n-1}}+c=\frac{1}{2(1-n)(x^2+1)^{n-1}}+c, & \text{for } n=2,3,4,\ldots,\ x\in R.\end{cases}$$

---

$$\int\frac{dx}{x^2+4x+3}=\left[\begin{array}{c} x^2+4x+3=(x+2)^2-1 \\ x+2=t,\ \ dx=dt\end{array}\right]=\int\frac{dt}{t^2-1}=\int\frac{dt}{(t-1)(t+1)}=\frac{1}{2}\int\left[\frac{1}{t-1}-\frac{1}{t+1}\right]dx$$
$$=\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|+c=\frac{1}{2}\ln\left|\frac{x+1}{x+3}\right|+c,\ \text{for } x\in R-\{-1,-3\}.$$

$$\int\frac{dx}{x^2+4x+3}=\left[\begin{array}{c} x^2+4x+3=(x+3)(x+1) \\ \frac{1}{(x+3)(x+1)}=\frac{1}{2}\left[\frac{1}{x+1}-\frac{1}{x+3}\right]\end{array}\right]=\frac{1}{2}\int\left[\frac{1}{x+1}-\frac{1}{x+3}\right]dx$$
$$=\frac{1}{2}\left[\ln|x+1|-\ln|x+3|\right]=\frac{1}{2}\ln\left|\frac{x+1}{x+3}\right|+c,\ \text{for } x\in R-\{-1,-3\}.$$

---

$$\int\frac{dx}{x^2+4x+2}=\left[\begin{array}{c} x^2+4x+2=(x+2)^2-2 \\ x+2=t,\ \ dx=dt\end{array}\right]=\int\frac{dt}{t^2-2}=\frac{1}{2}\ln\left|\frac{t-\sqrt{2}}{t+\sqrt{2}}\right|+c=\frac{1}{2}\ln\left|\frac{x+2-\sqrt{2}}{x+2+\sqrt{2}}\right|+c,$$
for $x\in R-\{-2\pm\sqrt{2}\}$.

$$\int\frac{dx}{x^2+4x+2}=\int\frac{dx}{(x+2)^2-2}=\frac{1}{2}\ln\left|\frac{x+2-\sqrt{2}}{x+2+\sqrt{2}}\right|+c,\ \text{for } x\in R-\{-2\pm\sqrt{2}\}.$$

---

$$\int \frac{dx}{x^2+4x+5} = \left[ \begin{matrix} x^2+4x+5=(x+2)^2+1 \\ x+2=t, \ \ dx=dt \end{matrix} \right] = \int \frac{dt}{t^2+1} = \operatorname{arctg} t + c = \operatorname{arctg}(x+2) + c, \text{ for } x \in R.$$

$$\int \frac{dx}{x^2+4x+5} = \int \frac{dx}{(x+2)^2+1} = \operatorname{arctg}(x+2) + c, \text{ for } x \in R.$$

---

$$\int \frac{dx}{x^2+4x+6} = \left[ \begin{matrix} x^2+4x+6=(x+2)^2+2 \\ x+2=t, \ \ dx=dt \end{matrix} \right] = \int \frac{dt}{t^2+2} = \frac{1}{\sqrt{2}} \operatorname{arctg} \frac{t}{\sqrt{2}} + c = \frac{1}{\sqrt{2}} \operatorname{arctg} \frac{x+2}{\sqrt{2}} + c,$$
for $x \in R$.

$$\int \frac{dx}{x^2+4x+6} = \int \frac{dt}{(x+2)^2+2} = \frac{1}{\sqrt{2}} \operatorname{arctg} \frac{x+2}{\sqrt{2}} + c, \text{ for } x \in R.$$

---

$$\int \frac{dx}{(x^2+4x+3)^2} = \left[ \begin{matrix} x^2+4x+3=(x+2)^2-1=t^2-1 \\ x+2=t, \ \ dx=dt \end{matrix} \right] = \int \frac{dt}{(t^2-1)^2} = -\int \frac{-\,dt}{(t^2-1)^2}$$
$$= -\int \frac{(t^2-1-t^2)\,dt}{(t^2-1)^2} = -\int \frac{dt}{t^2-1} + \int \frac{t^2\,dt}{(t^2-1)^2} = -\int \frac{dt}{t^2-1} + \frac{1}{2}\int \frac{t\cdot 2t\,dt}{(t^2-1)^2}$$
$$= \left[ \begin{matrix} u=t \\ v'=\frac{2t}{(t^2-1)^2} \end{matrix} \middle| \begin{matrix} u'=1 \\ v=\frac{(t^2-1)^{1-2}}{1-2}=-\frac{1}{t^2-1} \end{matrix} \right] = -\int \frac{dt}{t^2-1} + \frac{1}{2}\left[ -\frac{t}{t^2-1} + \int \frac{dt}{t^2-1} \right]$$
$$= -\frac{1}{2}\int \frac{dt}{t^2-1} - \frac{1}{2}\frac{t}{t^2-1} = -\frac{1}{2}\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| - \frac{1}{2}\frac{t}{t^2-1} + c = -\frac{1}{4}\ln\left|\frac{x+2-1}{x+2+1}\right| - \frac{1}{2}\frac{x+2}{x^2+4x+3} + c$$
$$= -\frac{1}{4}\ln\left|\frac{x+1}{x+3}\right| - \frac{1}{2}\frac{x+2}{x^2+4x+3} + c, \text{ for } x \in R - \{-1, -3\}.$$

---

$$\int \frac{dx}{(x^2+4x+2)^2} = \left[ \begin{matrix} x^2+4x+2=(x+2)^2-2=t^2-2 \\ x+2=t, \ \ dx=dt \end{matrix} \right] = \int \frac{dt}{(t^2-2)^2} = -\frac{1}{2}\int \frac{-2\,dt}{(t^2-2)^2}$$
$$= -\frac{1}{2}\int \frac{(t^2-2-t^2)\,dt}{(t^2-2)^2} = -\frac{1}{2}\int \frac{dt}{t^2-2} + \frac{1}{2}\int \frac{t^2\,dt}{(t^2-2)^2} = -\frac{1}{2}\int \frac{dt}{t^2-2} + \frac{1}{4}\int \frac{t\cdot 2t\,dt}{(t^2-2)^2}$$
$$= \left[ \begin{matrix} u=t \\ v'=\frac{2t}{(t^2-2)^2} \end{matrix} \middle| \begin{matrix} u'=1 \\ v=\frac{(t^2-2)^{1-2}}{1-2}=-\frac{1}{t^2-2} \end{matrix} \right] = -\frac{1}{2}\int \frac{dt}{t^2-2} + \frac{1}{4}\left[ -\frac{t}{t^2-2} + \int \frac{dt}{t^2-2} \right]$$
$$= -\frac{1}{4}\int \frac{dt}{t^2-2} - \frac{1}{4}\frac{t}{t^2-2} = -\frac{1}{4}\frac{1}{2\sqrt{2}}\ln\left|\frac{t-\sqrt{2}}{t+\sqrt{2}}\right| - \frac{1}{4}\frac{t}{t^2-2} + c$$
$$= -\frac{\sqrt{2}}{16}\ln\left|\frac{x+2-\sqrt{2}}{x+2+\sqrt{2}}\right| - \frac{1}{4}\frac{x+2}{x^2+4x+3} + c, \text{ for } x \in R - \{-2 \pm \sqrt{2}\}.$$

---

$$\int \frac{dx}{(x^2+4x+5)^2} = \left[ \begin{matrix} x^2+4x+5=(x+2)^2+1=t^2+1 \\ x+2=t, \ \ dx=dt \end{matrix} \right] = \int \frac{dt}{(t^2+1)^2} = \int \frac{(t^2+1-t^2)\,dt}{(t^2+1)^2}$$
$$= \int \frac{dt}{t^2+1} - \int \frac{t^2\,dt}{(t^2+1)^2} = \int \frac{dt}{t^2+1} - \frac{1}{2}\int \frac{t\cdot 2t\,dt}{(t^2+1)^2} = \left[ \begin{matrix} u=t \\ v'=\frac{2t}{(t^2+1)^2} \end{matrix} \middle| \begin{matrix} u'=1 \\ v=\frac{(t^2+1)^{1-2}}{1-2}=-\frac{1}{t^2+1} \end{matrix} \right]$$
$$= \int \frac{dt}{t^2+1} - \frac{1}{2}\left[ -\frac{t}{t^2+1} + \int \frac{dt}{t^2+1} \right] = \frac{1}{2}\int \frac{dt}{t^2+1} + \frac{1}{2}\frac{t}{t^2+1} = \frac{1}{2}\operatorname{arctg} t + \frac{1}{2}\frac{t}{t^2+1} + c$$
$$= \frac{1}{2}\operatorname{arctg}(x+2) + \frac{1}{2}\frac{x+2}{x^2+4x+5} + c, \text{ for } x \in R.$$

---

$$\int\frac{dx}{(x^2+4x+6)^2}=\left[\begin{matrix}x^2+4x+6=(x+2)^2+2=t^2+2\\ x+2=t,\ \ dx=dt\end{matrix}\right]=\int\frac{dt}{(t^2+2)^2}=\frac{1}{2}\int\frac{2\,dt}{(t^2+2)^2}=\frac{1}{2}\int\frac{(t^2+2-t^2)\,dt}{(t^2+2)^2}$$
$$=\frac{1}{2}\int\frac{dt}{t^2+2}-\frac{1}{2}\int\frac{t^2\,dt}{(t^2+2)^2}=\frac{1}{2}\int\frac{dt}{t^2+2}-\frac{1}{4}\int\frac{t\cdot 2t\,dt}{(t^2+2)^2}=\left[\begin{matrix}u=t & u'=1\\ v'=\frac{2t}{(t^2+2)^2} & v=\frac{(t^2+2)^{1-2}}{1-2}=-\frac{1}{t^2+2}\end{matrix}\right]$$
$$=\frac{1}{2}\int\frac{dt}{t^2+2}-\frac{1}{4}\left[-\frac{t}{t^2+2}+\int\frac{dt}{t^2+2}\right]=\frac{1}{4}\int\frac{dt}{t^2+2}+\frac{1}{4}\frac{t}{t^2+2}=\frac{1}{4}\frac{1}{\sqrt{2}}\operatorname{arctg}\frac{t}{\sqrt{2}}+\frac{1}{4}\frac{t}{t^2+2}+c$$
$$=\frac{\sqrt{2}}{8}\operatorname{arctg}\frac{x+2}{\sqrt{2}}+\frac{1}{4}\frac{x+2}{x^2+4x+6}+c,\ \text{for } x\in R.$$

---

$$\int\frac{dx}{(x^2+4x+2)^3}=\left[\begin{matrix}x^2+4x+2=(x+2)^2-2=t^2-2\\ x+2=t,\ \ dx=dt\end{matrix}\right]=\int\frac{dt}{(t^2-2)^3}=-\frac{1}{2}\int\frac{-2\,dt}{(t^2-2)^3}$$
$$=-\frac{1}{2}\int\frac{(t^2-2-t^2)\,dt}{(t^2-2)^3}=-\frac{1}{2}\int\frac{dt}{(t^2-2)^2}+\frac{1}{2}\int\frac{t^2\,dt}{(t^2-2)^3}=-\frac{1}{2}\int\frac{dt}{(t^2-2)^2}+\frac{1}{4}\int\frac{t\cdot 2t\,dt}{(t^2-2)^3}$$
$$=\left[\begin{matrix}u=t & u'=1\\ v'=\frac{2t}{(t^2-2)^3} & v=\frac{(t^2-2)^{1-3}}{1-3}=-\frac{1}{2(t^2-2)^2}\end{matrix}\right]=-\frac{1}{2}\int\frac{dt}{(t^2-2)^2}+\frac{1}{4}\left[-\frac{t}{2(t^2-2)^2}+\int\frac{dt}{2(t^2-2)^2}\right]$$
$$=-\frac{3}{8}\int\frac{dt}{(t^2-2)^2}-\frac{1}{8}\frac{t}{(t^2-2)^2}=\frac{3}{16}\int\frac{-2\,dt}{(t^2-2)^2}-\frac{1}{8}\frac{t}{(t^2-2)^2}=\frac{3}{16}\int\frac{(t^2-2-t^2)\,dt}{(t^2-2)^2}-\frac{1}{8}\frac{t}{(t^2-2)^2}$$
$$=\frac{3}{16}\int\frac{dt}{t^2-2}-\frac{3}{16}\int\frac{t^2\,dt}{(t^2-2)^2}-\frac{1}{8}\frac{t}{(t^2-2)^2}=\frac{3}{16}\int\frac{dt}{t^2-2}-\frac{3}{32}\int\frac{t\cdot 2t\,dt}{(t^2-2)^2}-\frac{1}{8}\frac{t}{(t^2-2)^2}$$
$$=\left[\begin{matrix}u=t & u'=1\\ v'=\frac{2t}{(t^2-2)^2} & v=\frac{(t^2-2)^{1-2}}{1-2}=-\frac{1}{t^2-2}\end{matrix}\right]=\frac{3}{16}\int\frac{dt}{t^2-2}-\frac{3}{32}\left[-\frac{t}{t^2-2}+\int\frac{dt}{t^2-2}\right]-\frac{1}{8}\frac{t}{(t^2-2)^2}$$
$$=\frac{3}{32}\int\frac{dt}{t^2-2}+\frac{3}{32}\frac{t}{t^2-2}-\frac{1}{8}\frac{t}{(t^2-2)^2}=\frac{3}{32}\frac{1}{2\sqrt{2}}\ln\left|\frac{t-\sqrt{2}}{t+\sqrt{2}}\right|+\frac{3}{32}\frac{t}{t^2-2}-\frac{1}{8}\frac{t}{(t^2-2)^2}+c$$
$$=\frac{3\sqrt{2}}{128}\ln\left|\frac{x+2-\sqrt{2}}{x+2+\sqrt{2}}\right|+\frac{3}{32}\frac{x+2}{x^2+4x+3}-\frac{1}{8}\frac{x+2}{(x^2+4x+3)^2}+c,\ \text{for } x\in R-\{-2\pm\sqrt{2}\}.$$

---

$$\int\frac{dx}{(x^2+4x+6)^3}=\left[\begin{matrix}x^2+4x+6=(x+2)^2+2=t^2+2\\ x+2=t,\ \ dx=dt\end{matrix}\right]=\int\frac{dt}{(t^2+2)^3}=\frac{1}{2}\int\frac{2\,dt}{(t^2+2)^3}=\frac{1}{2}\int\frac{(t^2+2-t^2)\,dt}{(t^2+2)^3}$$
$$=\frac{1}{2}\int\frac{dt}{(t^2+2)^2}-\frac{1}{2}\int\frac{t^2\,dt}{(t^2+2)^3}=\frac{1}{2}\int\frac{dt}{(t^2+2)^2}-\frac{1}{4}\int\frac{t\cdot 2t\,dt}{(t^2+2)^3}$$
$$=\left[\begin{matrix}u=t & u'=1\\ v'=\frac{2t}{(t^2+2)^3} & v=\frac{(t^2+2)^{1-3}}{1-3}=-\frac{1}{2(t^2+2)^2}\end{matrix}\right]=\frac{1}{2}\int\frac{dt}{(t^2+2)^2}-\frac{1}{4}\left[-\frac{t}{2(t^2+2)^2}+\int\frac{dt}{2(t^2+2)^2}\right]$$
$$=\frac{3}{8}\int\frac{dt}{(t^2+2)^2}+\frac{1}{8}\frac{t}{(t^2+2)^2}=\frac{3}{16}\int\frac{2\,dt}{(t^2+2)^2}+\frac{1}{8}\frac{t}{(t^2+2)^2}=\frac{3}{16}\int\frac{(t^2+2-t^2)\,dt}{(t^2+2)^2}+\frac{1}{8}\frac{t}{(t^2+2)^2}$$
$$=\frac{3}{16}\int\frac{dt}{t^2+2}-\frac{3}{16}\int\frac{t^2\,dt}{(t^2+2)^2}+\frac{1}{8}\frac{t}{(t^2+2)^2}=\frac{3}{16}\int\frac{dt}{t^2+2}-\frac{3}{32}\int\frac{t\cdot 2t\,dt}{(t^2+2)^2}+\frac{1}{8}\frac{t}{(t^2+2)^2}$$
$$=\left[\begin{matrix}u=t & u'=1\\ v'=\frac{2t}{(t^2+2)^2} & v=\frac{(t^2+2)^{1-2}}{1-2}=-\frac{1}{t^2+2}\end{matrix}\right]=\frac{3}{16}\int\frac{dt}{t^2+2}-\frac{3}{32}\left[-\frac{t}{t^2+2}+\int\frac{dt}{t^2+2}\right]+\frac{1}{8}\frac{t}{(t^2+2)^2}$$
$$=\frac{3}{32}\int\frac{dt}{t^2+2}+\frac{3}{32}\frac{t}{t^2+2}+\frac{1}{8}\frac{t}{(t^2+2)^2}=\frac{3}{32}\frac{1}{\sqrt{2}}\operatorname{arctg}\frac{t}{\sqrt{2}}+\frac{3}{32}\frac{t}{t^2+2}+\frac{1}{8}\frac{t}{(t^2+2)^2}$$
$$=\frac{3\sqrt{2}}{64}\operatorname{arctg}\frac{x+2}{\sqrt{2}}+\frac{3}{32}\frac{x+2}{x^2+4x+3}+\frac{1}{8}\frac{x+2}{(x^2+4x+3)^2}+c,\ \text{for } x\in R.$$

---

# 6 Examples

$$\int\left[\sqrt{x^3}-\frac{1}{\sqrt{x}}\right]dx=\int\left[x^{\frac{3}{2}}-x^{-\frac{1}{2}}\right]dx=\tfrac{2}{5}x^{\frac{5}{2}}-2x^{\frac{1}{2}}+c=\frac{2\sqrt{x^5}}{5}-2\sqrt{x}+c,\text{ for } x>0.$$

$$\int x(x-a)(x-b)\,dx=\int\left[x^3-(a+b)x^2+abx\right]dx=\tfrac{1}{4}x^4-\tfrac{a+b}{3}x^3+\tfrac{ab}{2}x^2+c,$$
for $a,b\in R$, $x\in R$.

$$\int e^{ax}\,dx=\frac{1}{a}\int a\,e^{ax}\,dx=\left[\begin{array}{c}ax=t\\ a\,dx=dt\end{array}\right]=\frac{1}{a}\int e^t\,dt=\frac{e^t}{a}+c=\frac{e^{ax}}{a}+c,\text{ for } a\in R-\{0\},\ x\in R.$$

$$\int\sinh^2x\,dx=\int\frac{(e^x-e^{-x})^2}{4}\,dx=\frac{1}{4}\int\left[e^{2x}-2e^xe^{-x}+e^{-2x}\right]dx=\frac{1}{4}\left[\frac{e^{2x}}{2}-2x+\frac{e^{-2x}}{-2}\right]+c$$
$$=\frac{e^{2x}}{8}-\frac{e^{-2x}}{8}-\frac{x}{2}+c=\frac{\sinh 2x}{4}-\frac{x}{2}+c,\text{ for } x\in R.$$

$$\int\cosh^2x\,dx=\int\frac{(e^x+e^{-x})^2}{4}\,dx=\frac{1}{4}\int\left[e^{2x}+2e^xe^{-x}+e^{-2x}\right]dx$$
$$=\frac{1}{4}\left[\frac{e^{2x}}{2}+2x+\frac{e^{-2x}}{-2}\right]+c=\frac{e^{2x}}{8}-\frac{e^{-2x}}{8}+\frac{x}{2}+c=\frac{\sinh 2x}{4}+\frac{x}{2}+c,\text{ for } x\in R.$$

$$\int[\operatorname{tg}x+\operatorname{cotg}x]\,dx=\int\left[\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}\right]dx=\int\left[\frac{-(\cos x)'}{\cos x}+\frac{(\sin x)'}{\sin x}\right]dx$$
$$=-\ln|\cos x|+\ln|\sin x|+c=\ln\left|\frac{\sin x}{\cos x}\right|+c=\ln|\operatorname{tg}x|+c,\text{ for } x\in R-\left\{\frac{k\pi}{2}\,;\,k\in Z\right\}.$$

$$\int\frac{x\,dx}{x^2+a^2}=\frac{1}{2}\int\frac{2x\,dx}{x^2+a^2}=\frac{1}{2}\ln\left|x^2+a^2\right|+c=\frac{1}{2}\ln\left(x^2+a^2\right)+c,\text{ for } a\in R,\ a\neq0,\ x\in R.$$
$$\int\frac{x\,dx}{x^2-a^2}=\frac{1}{2}\int\frac{2x\,dx}{x^2-a^2}=\frac{1}{2}\ln\left|x^2-a^2\right|+c,\text{ for } a\in R,\ a\neq0,\ x\in R-\{\pm a\}.$$
$$\int\frac{dx}{x}=\int\frac{x\,dx}{x^2-0^2}=\frac{1}{2}\ln\left|x^2-0^2\right|+c=\frac{1}{2}\ln|x|^2+c=\ln|x|+c,\text{ for } a=0,\ x\in R-\{0\}.$$

$$\int\frac{x\,dx}{(x^2-a)^n}=\frac{1}{2}\int\frac{2x\,dx}{(x^2-a)^n}=\left[\begin{array}{c}x^2-a=t\\ 2x\,dx=dt\end{array}\right]=\frac{1}{2}\int\frac{dt}{t^n}=\frac{1}{2}\int t^{-n}\,dt=\frac{t^{1-n}}{2(1-n)}+c$$
$$=\frac{(x^2-a)^{1-n}}{2(1-n)}+c,\text{ for } n\in N,\ a\in R-\{0\},\ x\in R,\ x^2-a\neq0.$$

$$\int\frac{\ln\cos x}{\sin^2x}\,dx=\left[\begin{array}{l|l}u=\ln\cos x & u'=\frac{-\sin x}{\cos x}\\ v'=\frac{1}{\sin^2x} & v=-\operatorname{cotg}x=-\frac{\cos x}{\sin x}\end{array}\right]=-\operatorname{cotg}x\ln\cos x-\int dx$$
$$=-\operatorname{cotg}x\ln\cos x-x+c,\text{ for } x\left\langle-\frac{\pi}{2}+2k\pi\,;\,2k\pi\right)\cup\left(2k\pi\,;\,\frac{\pi}{2}+2k\pi\right\rangle,\ k\in Z.$$

$$\int\frac{dx}{x^6+1}=\left[\begin{array}{c}\frac{1}{x^6+1}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2-\sqrt{3}x+1}+\frac{Ex+F}{x^2+\sqrt{3}x+1}\\ A=0,\ B=\frac{1}{3},\ C=-\frac{1}{2\sqrt{3}},\ D=\frac{1}{3},\ E=\frac{1}{2\sqrt{3}},\ F=\frac{1}{3}\end{array}\right]$$

$$=\int\left[\frac{\frac{1}{3}}{x^2+1}+\frac{-\frac{x}{2\sqrt{3}}+\frac{1}{3}}{x^2-\sqrt{3}x+1}+\frac{\frac{x}{2\sqrt{3}}+\frac{1}{3}}{x^2+\sqrt{3}x+1}\right]dx=\frac{1}{3}\int\frac{dx}{x^2+1}+\frac{1}{2\sqrt{3}}\int\left[\frac{x+\frac{2\sqrt{3}}{3}}{x^2+\sqrt{3}x+1}-\frac{x-\frac{2\sqrt{3}}{3}}{x^2-\sqrt{3}x+1}\right]dx$$

$$=\frac{1}{3}\operatorname{arctg}x+\frac{1}{4\sqrt{3}}\int\left[\frac{2x+\frac{4\sqrt{3}}{3}}{x^2+\sqrt{3}x+1}-\frac{2x-\frac{4\sqrt{3}}{3}}{x^2-\sqrt{3}x+1}\right]dx$$

$$=\frac{1}{3}\operatorname{arctg}x+\frac{1}{4\sqrt{3}}\int\left[\frac{2x+\frac{4\sqrt{3}}{3}+\sqrt{3}-\sqrt{3}}{x^2+\sqrt{3}x+1}-\frac{2x-\frac{4\sqrt{3}}{3}-\sqrt{3}+\sqrt{3}}{x^2-\sqrt{3}x+1}\right]dx$$

$$=\left[\ x^2\pm\sqrt{3}x+1=\left(x\pm\frac{\sqrt{3}}{2}\right)^2+1-\frac{3}{4}=\left(x\pm\frac{\sqrt{3}}{2}\right)^2+\frac{1}{4}=\left(x\pm\frac{\sqrt{3}}{2}\right)^2+\left(\frac{1}{2}\right)^2>0\ \right]$$

$$=\frac{1}{3}\operatorname{arctg}x+\frac{1}{4\sqrt{3}}\int\left[\frac{2x+\sqrt{3}}{x^2+\sqrt{3}x+1}-\frac{2x-\sqrt{3}}{x^2-\sqrt{3}x+1}\right]dx+\frac{1}{4\sqrt{3}}\int\left[\frac{\frac{\sqrt{3}}{3}}{\left(x+\frac{\sqrt{3}}{2}\right)^2+\frac{1}{4}}-\frac{-\frac{\sqrt{3}}{3}}{\left(x-\frac{\sqrt{3}}{2}\right)^2+\frac{1}{4}}\right]dx$$

$$=\frac{1}{3}\operatorname{arctg}x+\frac{1}{4\sqrt{3}}\int\frac{(2x+\sqrt{3})\,dx}{x^2+\sqrt{3}x+1}-\frac{1}{4\sqrt{3}}\int\frac{(2x-\sqrt{3})\,dx}{x^2-\sqrt{3}x+1}+\frac{1}{12}\int\frac{dx}{\left(x+\frac{\sqrt{3}}{2}\right)^2+\frac{1}{4}}+\frac{1}{12}\int\frac{dx}{\left(x-\frac{\sqrt{3}}{2}\right)^2+\frac{1}{4}}$$

$$=\left[\ x+\frac{\sqrt{3}}{2}=t,\ dx=dt\ \middle|\ x-\frac{\sqrt{3}}{2}=u,\ dx=du\ \right]$$

$$=\frac{1}{3}\operatorname{arctg}x+\frac{1}{4\sqrt{3}}\ln\left|x^2+\sqrt{3}x+1\right|-\frac{1}{4\sqrt{3}}\ln\left|x^2-\sqrt{3}x+1\right|+\frac{1}{12}\int\frac{dt}{t^2+\frac{1}{4}}+\frac{1}{12}\int\frac{du}{u^2+\frac{1}{4}}$$

$$=\frac{1}{3}\operatorname{arctg}x+\frac{1}{4\sqrt{3}}\ln\left(x^2+\sqrt{3}x+1\right)-\frac{1}{4\sqrt{3}}\ln\left(x^2-\sqrt{3}x+1\right)+\frac{2}{12}\operatorname{arctg}2t+\frac{2}{12}\operatorname{arctg}2u+c$$

$$=\frac{1}{3}\operatorname{arctg}x+\frac{1}{4\sqrt{3}}\ln\frac{x^2+\sqrt{3}x+1}{x^2-\sqrt{3}x+1}+\frac{1}{6}\operatorname{arctg}\left(2x+\sqrt{3}\right)+\frac{1}{6}\operatorname{arctg}\left(2x-\sqrt{3}\right)+c,\ \text{for } x\in R.$$

---

$$\int\frac{x\,dx}{x^6+1}=\left[\begin{array}{c}x^2=t\\ 2x\,dx=dt\end{array}\right]=\frac{1}{2}\int\frac{dt}{t^3+1}=\frac{1}{2}\int\frac{dt}{(t+1)(t^2-t+1)}=\left[\begin{array}{c}\frac{1}{t^3+1}=\frac{A}{t+1}+\frac{Bt+C}{t^2-t+1}\\ A=\frac{1}{3},\ B=-\frac{1}{3},\ C=\frac{2}{3}\end{array}\right]$$

$$=\frac{1}{2}\int\left[\frac{\frac{1}{3}}{t+1}+\frac{-\frac{t}{3}+\frac{2}{3}}{t^2-t+1}\right]dt=\frac{1}{6}\int\frac{dt}{t+1}-\frac{1}{6}\int\frac{t-2}{t^2-t+1}\,dt=\frac{1}{6}\int\frac{dt}{t+1}-\frac{1}{12}\int\frac{2t-4}{t^2-t+1}\,dt$$

$$=\frac{1}{6}\int\frac{dt}{t+1}-\frac{1}{12}\int\frac{2t-1-3}{t^2-t+1}\,dt=\left[\ t^2-t+1=\left(t-\frac{1}{2}\right)^2+1-\frac{1}{4}=\left(t-\frac{1}{2}\right)^2+\frac{3}{4}>0\ \right]$$

$$=\frac{1}{6}\ln|t+1|-\frac{1}{12}\int\frac{2t-1}{t^2-t+1}\,dt+\frac{1}{12}\int\frac{3\,dt}{t^2-t+1}$$

$$=\frac{1}{12}\ln(t+1)^2-\frac{1}{12}\ln\left|t^2-t+1\right|+\frac{1}{4}\int\frac{dt}{\left(t-\frac{1}{2}\right)^2+\frac{3}{4}}=\left[\begin{array}{c}t-\frac{1}{2}=z\\ dt=dz\end{array}\right]$$

$$=\frac{1}{12}\ln(t+1)^2-\frac{1}{12}\ln\left(t^2-t+1\right)+\frac{1}{4}\int\frac{dz}{z^2+\frac{3}{4}}=\frac{1}{12}\ln\frac{(t+1)^2}{t^2-t+1}+\frac{1}{4}\frac{2}{\sqrt{3}}\operatorname{arctg}\frac{2z}{\sqrt{3}}+c_1$$

$$=\frac{1}{12}\ln\frac{t^2+2t+1}{t^2-t+1}+\frac{1}{2\sqrt{3}}\operatorname{arctg}\frac{2t-1}{\sqrt{3}}+c_1=\frac{1}{12}\ln\frac{x^4+2x^2+1}{x^4-x^2+1}+\frac{\sqrt{3}}{6}\operatorname{arctg}\frac{2x^2-1}{\sqrt{3}}+c_1,\ \text{for } x\in R.$$

---

$$\int\frac{x\,dx}{x^6+1}=\left[\frac{x}{x^6+1}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2-\sqrt{3}x+1}+\frac{Ex+F}{x^2+\sqrt{3}x+1}\ \middle|\ A=\frac{1}{3},B=0,C=-\frac{1}{6},D=\frac{\sqrt{3}}{6},E=-\frac{1}{6},F=-\frac{\sqrt{3}}{6}\right]$$

$$=\int\left[\frac{\frac{x}{3}}{x^2+1}+\frac{-\frac{x}{6}+\frac{\sqrt{3}}{6}}{x^2-\sqrt{3}x+1}+\frac{-\frac{x}{6}-\frac{\sqrt{3}}{6}}{x^2+\sqrt{3}x+1}\right]dx=\frac{1}{6}\int\frac{2x\,dx}{x^2+1}-\frac{1}{12}\int\left[\frac{2x-2\sqrt{3}}{x^2-\sqrt{3}x+1}+\frac{2x+2\sqrt{3}}{x^2+\sqrt{3}x+1}\right]dx$$

$$=\frac{1}{6}\ln\left|x^2+1\right|-\frac{1}{12}\int\left[\frac{2x-\sqrt{3}-\sqrt{3}}{x^2-\sqrt{3}x+1}+\frac{2x+\sqrt{3}+\sqrt{3}}{x^2+\sqrt{3}x+1}\right]dx=\left[\ x^2\pm\sqrt{3}x+1=\left(x\pm\frac{\sqrt{3}}{2}\right)^2+\frac{1}{4}>0\ \right]$$

$$=\frac{1}{12}\ln\left(x^2+1\right)^2-\frac{1}{12}\int\left[\frac{2x-\sqrt{3}}{x^2-\sqrt{3}x+1}+\frac{2x+\sqrt{3}}{x^2+\sqrt{3}x+1}\right]dx+\frac{1}{12}\int\left[\frac{\sqrt{3}}{\left(x-\frac{\sqrt{3}}{2}\right)^2+\frac{1}{4}}-\frac{\sqrt{3}}{\left(x+\frac{\sqrt{3}}{2}\right)^2+\frac{1}{4}}\right]dx$$

$$=\left[\ x-\frac{\sqrt{3}}{2}=t,\ dx=dt\ \middle|\ x+\frac{\sqrt{3}}{2}=u,\ dx=du\ \right]$$

$$= \tfrac{1}{12}\ln\left(x^4+2x^2+1\right) - \tfrac{1}{12}\ln\left|x^2-\sqrt{3}x+1\right| - \tfrac{1}{12}\ln\left|x^2+\sqrt{3}x+1\right| + \tfrac{1}{12}\int\frac{\sqrt{3}\,\mathrm{d}t}{t^2+\frac{1}{4}} - \tfrac{1}{12}\int\frac{\sqrt{3}\,\mathrm{d}u}{u^2+\frac{1}{4}}$$
$$= \left[\ \left|x^2-\sqrt{3}x+1\right|\cdot\left|x^2+\sqrt{3}x+1\right| = \left(x^2-\sqrt{3}x+1\right)\left(x^2+\sqrt{3}x+1\right) = x^4-x^2+1>0\ \right]$$
$$= \tfrac{1}{12}\ln\frac{x^4+2x^2+1}{x^4-x^2+1} + \tfrac{2\sqrt{3}}{12}\operatorname{arctg}2t - \tfrac{2\sqrt{3}}{12}\operatorname{arctg}2u + c_2$$
$$= \tfrac{1}{12}\ln\frac{x^4+2x^2+1}{x^4-x^2+1} + \tfrac{\sqrt{3}}{6}\operatorname{arctg}\left(2x-\sqrt{3}\right) - \tfrac{\sqrt{3}}{6}\operatorname{arctg}\left(2x+\sqrt{3}\right) + c_2, \text{ for } x\in R.$$

---

$$\int\frac{x\,\mathrm{d}x}{x^6+1} = \tfrac{1}{12}\ln\frac{x^4+2x^2+1}{x^4-x^2+1} + \tfrac{\sqrt{3}}{6}\operatorname{arctg}\frac{2x^2-1}{\sqrt{3}} + c_1$$
$$= \tfrac{1}{12}\ln\frac{x^4+2x^2+1}{x^4-x^2+1} + \tfrac{\sqrt{3}}{6}\operatorname{arctg}\left(2x-\sqrt{3}\right) - \tfrac{\sqrt{3}}{6}\operatorname{arctg}\left(2x+\sqrt{3}\right) + c_2, \text{ for } x\in R.$$

$$\tfrac{1}{12}\ln\frac{x^4+2x^2+1}{x^4-x^2+1} + \tfrac{\sqrt{3}}{6}\operatorname{arctg}\frac{2x^2-1}{\sqrt{3}} - \left[\tfrac{1}{12}\ln\frac{x^4+2x^2+1}{x^4-x^2+1} + \tfrac{\sqrt{3}}{6}\operatorname{arctg}\left(2x-\sqrt{3}\right) - \tfrac{\sqrt{3}}{6}\operatorname{arctg}\left(2x+\sqrt{3}\right)\right]$$
$$= \tfrac{\sqrt{3}}{6}\operatorname{arctg}\frac{2x^2-1}{\sqrt{3}} - \tfrac{\sqrt{3}}{6}\operatorname{arctg}\left(2x-\sqrt{3}\right) + \tfrac{\sqrt{3}}{6}\operatorname{arctg}\left(2x+\sqrt{3}\right)$$
$$= \tfrac{\sqrt{3}}{6}\left[\operatorname{arctg}\frac{2x^2-1}{\sqrt{3}} - \operatorname{arctg}\left(2x-\sqrt{3}\right) + \operatorname{arctg}\left(2x+\sqrt{3}\right)\right] = \left[\begin{matrix}\operatorname{arctg}\alpha = \frac{\pi}{2} - \operatorname{arccotg}\alpha \\ \text{for } \alpha\in R\end{matrix}\right]$$
$$= \tfrac{\sqrt{3}}{6}\left[\operatorname{arctg}\frac{2x^2-1}{\sqrt{3}} - \tfrac{\pi}{2} + \operatorname{arccotg}\left(2x-\sqrt{3}\right) + \tfrac{\pi}{2} - \operatorname{arccotg}\left(2x+\sqrt{3}\right)\right]$$
$$= \tfrac{\sqrt{3}}{6}\left[\operatorname{arctg}\frac{2x^2-1}{\sqrt{3}} + \operatorname{arccotg}\left(2x-\sqrt{3}\right) - \operatorname{arccotg}\left(2x+\sqrt{3}\right)\right]$$
$$= \left[\ \operatorname{arccotg}\alpha - \operatorname{arccotg}\beta = \operatorname{arccotg}\frac{\alpha\beta+1}{\beta-\alpha},\ \text{ for } \alpha,\beta\in R,\ \alpha\neq\beta\ \right]$$
$$= \tfrac{\sqrt{3}}{6}\left[\operatorname{arctg}\frac{2x^2-1}{\sqrt{3}} + \operatorname{arccotg}\frac{\left(2x-\sqrt{3}\right)\left(2x+\sqrt{3}\right)+1}{\left(2x+\sqrt{3}\right)-\left(2x-\sqrt{3}\right)}\right]$$
$$= \tfrac{\sqrt{3}}{6}\left[\operatorname{arctg}\frac{2x^2-1}{\sqrt{3}} + \operatorname{arccotg}\frac{4x^2-3+1}{2\sqrt{3}}\right] = \tfrac{\sqrt{3}}{6}\left[\operatorname{arctg}\frac{2x^2-1}{\sqrt{3}} + \operatorname{arccotg}\frac{2x^2-1}{\sqrt{3}}\right]$$
$$= \left[\ \operatorname{arctg}\alpha + \operatorname{arctg}\alpha = \tfrac{\pi}{2},\ \text{ for } \alpha\in R\ \right] = \tfrac{\sqrt{3}}{6}\,\tfrac{\pi}{2} = \ \text{const.}, \text{ for } x\in R.$$

---

$$\int\frac{x^2\,\mathrm{d}x}{x^6+1} = \left[\ \frac{x^2}{x^6+1} = \frac{Ax+B}{x^2-\sqrt{3}x+1} + \frac{Cx+D}{x^2+\sqrt{3}x+1} + \frac{Ex+F}{x^2+1}\ \middle|\ A=0,\ B=\tfrac{1}{6},\ C=0,\ D=\tfrac{1}{6},\ E=0,\ F=-\tfrac{1}{3}\ \right]$$
$$= \int\left[\frac{\frac{1}{6}}{x^2-\sqrt{3}x+1} + \frac{\frac{1}{6}}{x^2+\sqrt{3}x+1} + \frac{-\frac{1}{3}}{x^2+1}\right]\mathrm{d}x = \tfrac{1}{6}\int\frac{\mathrm{d}x}{x^2-\sqrt{3}x+1} + \tfrac{1}{6}\int\frac{\mathrm{d}x}{x^2+\sqrt{3}x+1} - \tfrac{1}{3}\int\frac{\mathrm{d}x}{x^2+1}$$
$$= \left[\ x^2\pm\sqrt{3}x+1 = \left(x\pm\tfrac{\sqrt{3}}{2}\right)^2 + \tfrac{1}{4} > 0\ \right] = \tfrac{1}{6}\int\frac{\mathrm{d}x}{\left(x-\frac{\sqrt{3}}{2}\right)^2+\frac{1}{4}} + \tfrac{1}{6}\int\frac{\mathrm{d}x}{\left(x+\frac{\sqrt{3}}{2}\right)^2+\frac{1}{4}} - \tfrac{1}{3}\operatorname{arctg}x$$
$$= \left[\ x-\tfrac{\sqrt{3}}{2} = t,\ \mathrm{d}x = \mathrm{d}t\ \middle|\ x+\tfrac{\sqrt{3}}{2} = u,\ \mathrm{d}x = \mathrm{d}u\ \right] = \tfrac{1}{6}\int\frac{\mathrm{d}t}{t^2+\frac{1}{4}} + \tfrac{1}{6}\int\frac{\mathrm{d}u}{u^2+\frac{1}{4}} - \tfrac{1}{3}\operatorname{arctg}x$$
$$= \tfrac{2}{6}\operatorname{arctg}2t + \tfrac{2}{6}\operatorname{arctg}2u - \tfrac{1}{3}\operatorname{arctg}x + c_2$$
$$= \tfrac{1}{3}\operatorname{arctg}\left(2x-\sqrt{3}\right) + \tfrac{1}{3}\operatorname{arctg}\left(2x+\sqrt{3}\right) - \tfrac{1}{3}\operatorname{arctg}x + c_1, \text{ for } x\in R.$$

---

$$\int\frac{x^2\,\mathrm{d}x}{x^6+1} = \tfrac{1}{3}\int\frac{3x^2\,\mathrm{d}x}{x^6+1} = \left[\begin{matrix}x^3 = t\\ 3x^2\mathrm{d}x = \mathrm{d}t\end{matrix}\right] = \tfrac{1}{3}\int\frac{\mathrm{d}t}{t^2+1} = \tfrac{1}{3}\operatorname{arctg}t + c_1 = \tfrac{1}{3}\operatorname{arctg}x^3 + c_2, \text{ for } x\in R.$$

---

$$\int\frac{x^2\,\mathrm{d}x}{x^6+1} = \tfrac{1}{3}\operatorname{arctg}\left(2x-\sqrt{3}\right) + \tfrac{1}{3}\operatorname{arctg}\left(2x+\sqrt{3}\right) - \tfrac{1}{3}\operatorname{arctg}x + c_1, \text{ for } x\in R,$$
$$= \tfrac{1}{3}\operatorname{arctg}x^3 + c_2, \text{ for } x\in R.$$

$$\tfrac{1}{3}\,\mathrm{arctg}\,(2x-\sqrt{3})+\tfrac{1}{3}\,\mathrm{arctg}\,(2x+\sqrt{3})-\tfrac{1}{3}\,\mathrm{arctg}\,x-\tfrac{1}{3}\,\mathrm{arctg}\,x^3=\left[\begin{array}{c}\mathrm{arctg}\,\alpha=\frac{\pi}{2}-\mathrm{arccotg}\,\alpha\\ \text{for }\alpha\in R\end{array}\right]$$
$$=\tfrac{\pi}{2}-\tfrac{1}{3}\,\mathrm{arccotg}\,(2x-\sqrt{3})+\tfrac{\pi}{2}-\tfrac{1}{3}\,\mathrm{arccotg}\,(2x+\sqrt{3})-\tfrac{\pi}{2}+\tfrac{1}{3}\,\mathrm{arccotg}\,x-\tfrac{\pi}{2}+\tfrac{1}{3}\,\mathrm{arccotg}\,x^3$$
$$=\tfrac{1}{3}\left[\mathrm{arccotg}\,x+\mathrm{arccotg}\,x^3\right]-\tfrac{1}{3}\left[\mathrm{arccotg}\,(2x-\sqrt{3})+\mathrm{arccotg}\,(2x+\sqrt{3})\right]$$
$$=\left[\ \mathrm{arccotg}\,\alpha+\mathrm{arccotg}\,\beta=\mathrm{arccotg}\,\tfrac{\alpha\beta-1}{\alpha+\beta},\ \text{for }\alpha,\beta\in R,\ \alpha\neq-\beta\ \right]$$
$$=\tfrac{1}{3}\,\mathrm{arccotg}\,\tfrac{x\cdot x^3-1}{x+x^3}-\tfrac{1}{3}\,\mathrm{arccotg}\,\tfrac{(2x-\sqrt{3})(2x+\sqrt{3})-1}{(2x+\sqrt{3})+(2x-\sqrt{3})}=\tfrac{1}{3}\,\mathrm{arccotg}\,\tfrac{x^4-1}{x^3+x}-\tfrac{1}{3}\,\mathrm{arccotg}\,\tfrac{4x^2-4}{4x}$$
$$=\tfrac{1}{3}\,\mathrm{arccotg}\,\tfrac{(x^2-1)(x^2+1)}{x(x^2+1)}-\tfrac{1}{3}\,\mathrm{arccotg}\,\tfrac{x^2-1}{x}=\tfrac{1}{3}\,\mathrm{arccotg}\,\tfrac{x^2-1}{x}-\tfrac{1}{3}\,\mathrm{arccotg}\,\tfrac{x^2-1}{x}$$
$$=0=\ \text{const., for } x\in R.$$

---

$$\int\frac{x^3\,\mathrm{d}x}{x^6+1}=\frac{1}{2}\int\frac{2x\cdot x^2}{x^6+1}\,\mathrm{d}x=\left[\begin{array}{c}x^2=t\\ 2x\,\mathrm{d}x=\mathrm{d}t\end{array}\right]=\frac{1}{2}\int\frac{t\,\mathrm{d}t}{t^3+1}=\frac{1}{2}\int\frac{t\,\mathrm{d}t}{(t+1)(t^2-t+1)}$$
$$=\left[\ \tfrac{t}{t^3+1}=\tfrac{A}{t+1}+\tfrac{Bt+C}{t^2-t+1}\ \Big|\ A=-\tfrac{1}{3},\ B=\tfrac{1}{3},\ C=\tfrac{1}{3}\ \right]=\frac{1}{2}\int\left[\frac{-\frac{1}{3}}{t+1}+\frac{\frac{t}{3}+\frac{1}{3}}{t^2-t+1}\right]\mathrm{d}t$$
$$=\frac{1}{6}\int\frac{t+1}{t^2-t+1}\,\mathrm{d}t-\frac{1}{6}\int\frac{\mathrm{d}t}{t+1}=\frac{1}{12}\int\frac{2t+2}{t^2-t+1}\,\mathrm{d}t-\frac{1}{6}\int\frac{\mathrm{d}t}{t+1}=\frac{1}{12}\int\frac{2t-1+3}{t^2-t+1}\,\mathrm{d}t-\frac{1}{6}\int\frac{\mathrm{d}t}{t+1}$$
$$=\left[\ t^2-t+1=(t-\tfrac{1}{2})^2+1-\tfrac{1}{4}=(t-\tfrac{1}{2})^2+\tfrac{3}{4}>0\ \right]=\frac{1}{12}\int\frac{2t-1}{t^2-t+1}\,\mathrm{d}t+\frac{1}{4}\int\frac{\mathrm{d}t}{(t-\frac{1}{2})^2+\frac{3}{4}}-\frac{1}{6}\int\frac{\mathrm{d}t}{t+1}$$
$$=\left[\begin{array}{c}z=t-\frac{1}{2}\\ \mathrm{d}z=\mathrm{d}t\end{array}\right]=\frac{1}{12}\ln\left(t^2-t+1\right)+\frac{1}{4}\int\frac{\mathrm{d}z}{z^2+\frac{3}{4}}-\frac{1}{6}\ln|t+1|$$
$$=\frac{1}{12}\ln\left(t^2-t+1\right)+\frac{1}{4}\frac{2}{\sqrt{3}}\,\mathrm{arctg}\,\frac{2z}{\sqrt{3}}-\frac{1}{12}\ln(t+1)^2=\frac{1}{12}\ln\frac{t^2-t+1}{(t+1)^2}+\frac{1}{2\sqrt{3}}\,\mathrm{arctg}\,\frac{2t-1}{\sqrt{3}}+c$$
$$=\frac{1}{12}\ln\frac{x^4-x^2+1}{(x^2+1)^2}+\frac{1}{2\sqrt{3}}\,\mathrm{arctg}\,\frac{2x^2-1}{\sqrt{3}}+c,\ \text{for } x\in R.$$

---

$$\int\frac{x^4\,\mathrm{d}x}{x^6+1}=\left[\begin{array}{c}\frac{x^4}{x^6+1}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2-\sqrt{3}x+1}+\frac{Ex+F}{x^2+\sqrt{3}x+1}\\ A=0,\ B=\frac{1}{3},\ C=\frac{1}{2\sqrt{3}},\ D=-\frac{1}{6},\ E=-\frac{1}{2\sqrt{3}},\ F=-\frac{1}{6}\end{array}\right]$$
$$=\int\left[\frac{\frac{1}{3}}{x^2+1}+\frac{\frac{x}{2\sqrt{3}}-\frac{1}{6}}{x^2-\sqrt{3}x+1}+\frac{-\frac{x}{2\sqrt{3}}-\frac{1}{6}}{x^2+\sqrt{3}x+1}\right]\mathrm{d}x=\frac{1}{3}\int\frac{\mathrm{d}x}{x^2+1}+\frac{1}{2\sqrt{3}}\int\left[\frac{x-\frac{\sqrt{3}}{3}}{x^2-\sqrt{3}x+1}-\frac{x+\frac{\sqrt{3}}{3}}{x^2+\sqrt{3}x+1}\right]\mathrm{d}x$$
$$=\frac{1}{3}\,\mathrm{arctg}\,x+\frac{1}{4\sqrt{3}}\int\left[\frac{2x-\frac{2\sqrt{3}}{3}+\sqrt{3}-\sqrt{3}}{x^2-\sqrt{3}x+1}-\frac{2x+\frac{2\sqrt{3}}{3}-\sqrt{3}+\sqrt{3}}{x^2+\sqrt{3}x+1}\right]\mathrm{d}x$$
$$=\left[\ x^2\pm\sqrt{3}x+1=\left(x\pm\tfrac{\sqrt{3}}{2}\right)^2+1-\tfrac{3}{4}=\left(x\pm\tfrac{\sqrt{3}}{2}\right)^2+\tfrac{1}{4}>0\ \right]$$
$$=\frac{1}{3}\,\mathrm{arctg}\,x+\frac{1}{4\sqrt{3}}\int\left[\frac{2x-\sqrt{3}}{x^2-\sqrt{3}x+1}-\frac{2x+\sqrt{3}}{x^2+\sqrt{3}x+1}\right]\mathrm{d}x+\frac{1}{4\sqrt{3}}\int\left[\frac{\frac{\sqrt{3}}{3}}{\left(x-\frac{\sqrt{3}}{2}\right)^2+\frac{1}{4}}-\frac{-\frac{\sqrt{3}}{3}}{\left(x+\frac{\sqrt{3}}{2}\right)^2+\frac{1}{4}}\right]\mathrm{d}x$$
$$=\left[\ x-\tfrac{\sqrt{3}}{2}=t,\ \mathrm{d}x=\mathrm{d}t\ \Big|\ x+\tfrac{\sqrt{3}}{2}=u,\ \mathrm{d}x=\mathrm{d}u\ \right]$$
$$=\frac{1}{3}\,\mathrm{arctg}\,x+\frac{1}{4\sqrt{3}}\ln\left|x^2-\sqrt{3}x+1\right|-\frac{1}{4\sqrt{3}}\ln\left|x^2+\sqrt{3}x+1\right|+\frac{1}{12}\int\frac{\mathrm{d}t}{t^2+\frac{1}{4}}+\frac{1}{12}\int\frac{\mathrm{d}u}{u^2+\frac{1}{4}}$$
$$=\frac{1}{3}\,\mathrm{arctg}\,x+\frac{1}{4\sqrt{3}}\ln\frac{x^2-\sqrt{3}x+1}{x^2+\sqrt{3}x+1}+\frac{2}{12}\,\mathrm{arctg}\,2t+\frac{2}{12}\,\mathrm{arctg}\,2u+c$$
$$=\frac{1}{3}\,\mathrm{arctg}\,x+\frac{1}{4\sqrt{3}}\ln\frac{x^2-\sqrt{3}x+1}{x^2+\sqrt{3}x+1}+\frac{1}{6}\,\mathrm{arctg}\,(2x-\sqrt{3})+\frac{1}{6}\,\mathrm{arctg}\,(2x+\sqrt{3})+c,\ \text{for } x\in R.$$

---

$$\int\frac{x^5\,\mathrm{d}x}{x^6+1}=\frac{1}{6}\int\frac{6x^5}{x^6+1}\,\mathrm{d}x=\left[\begin{array}{c}x^6=t\\ 6x^5\,\mathrm{d}x=\mathrm{d}t\end{array}\right]=\frac{1}{3}\int\frac{\mathrm{d}t}{t+1}=\frac{1}{6}\ln|t+1|+c=\frac{1}{6}\ln\left|x^6+1\right|+c$$
$$=\frac{1}{6}\ln\left(x^6+1\right)+c,\ \text{for } x\in R.$$

---

$$\int \frac{x^6\,\mathrm{d}x}{x^6+1} = \int \frac{x^6+1-1}{x^6+1}\,\mathrm{d}x = \int \Big[1 - \frac{1}{x^6+1}\Big]\,\mathrm{d}x = \int \mathrm{d}x - \int \frac{\mathrm{d}x}{x^6+1}$$
$$= x - \tfrac{1}{3}\operatorname{arctg} x - \tfrac{1}{4\sqrt{3}} \ln \frac{x^2+\sqrt{3}x+1}{x^2-\sqrt{3}x+1} - \tfrac{1}{6}\operatorname{arctg}\left(2x+\sqrt{3}\right) - \tfrac{1}{6}\operatorname{arctg}\left(2x-\sqrt{3}\right) + c, \text{ for } x \in R.$$

---

$$\int \frac{\mathrm{d}x}{x\ln x} = \int \frac{\frac{1}{x}}{\ln x}\,\mathrm{d}x = \int \frac{[\ln x]'}{\ln x}\,\mathrm{d}x = \ln|\ln x| + c, \text{ for } x > 0.$$

---

$$\int \frac{\ln x}{\sqrt{x}}\,\mathrm{d}x = \left[\begin{array}{l|l} u = \ln x & u' = \frac{1}{x} \\ v' = x^{-\frac{1}{2}} & v = 2x^{\frac{1}{2}} \end{array}\right] = 2\sqrt{x}\ln x - 2\int \frac{x^{\frac{1}{2}}}{x}\,\mathrm{d}x = 2\sqrt{x}\ln x - 2\int x^{-\frac{1}{2}}\,\mathrm{d}x$$
$$= 2\sqrt{x}\ln x - 2\cdot 2x^{\frac{1}{2}} + c = 2\sqrt{x}\ln x - 4\sqrt{x} + c, \text{ for } x > 0.$$

---

$$\int \frac{\mathrm{d}x}{\sqrt{x(1-x)}} = \left[\begin{array}{c} x(1-x) = x - x^2 > 0,\ x \in (0\,;1) \\ t = \frac{1}{x},\ x = \frac{1}{t},\ \mathrm{d}x = -\frac{\mathrm{d}t}{t^2},\ t > 1 \end{array}\right] = -\int \frac{1}{\sqrt{\frac{1}{t}\left(1-\frac{1}{t}\right)}}\frac{\mathrm{d}t}{t^2} = -\int \frac{\mathrm{d}t}{t\sqrt{t-1}} = \cdots \text{☹}$$

---

$$\int \frac{\mathrm{d}x}{\sqrt{x(1-x)}} = \left[\begin{array}{c} x(x-1) = x^2 - x = (x-\frac{1}{2})^2 - \frac{1}{4},\ x \in (0\,;1) \\ x - \frac{1}{2} = t,\ \mathrm{d}x = \mathrm{d}t,\ t \in \left(-\frac{1}{2}\,;\frac{1}{2}\right) \end{array}\right] = \int \frac{\mathrm{d}t}{\sqrt{\frac{1}{4}-t^2}} = \arcsin 2t + c$$
$$= \arcsin(2x-1) + c, \text{ for } x \in (0\,;1).$$

---

$$\int \frac{\mathrm{d}x}{\sqrt{x(1-x)}} = \left[\begin{array}{l} \boxed{\text{3rd ES}} \qquad x(1-x) = x - x^2 > 0,\ x \in (0\,;1) \\ t = \sqrt{\frac{x}{1-x}},\ t^2 = \frac{x}{1-x},\ t^2 - t^2x = x,\ x = \frac{t^2}{t^2+1},\ \mathrm{d}x = \frac{2t(t^2+1)-t^2\cdot 2t}{(t^2+1)^2}\,\mathrm{d}t = \frac{2t\,\mathrm{d}t}{(t^2+1)^2} \\ x(1-x) = \frac{t^2}{t^2+1}\left(1-\frac{t^2}{t^2+1}\right) = \frac{t^2(t^2+1-t^2)}{(t^2+1)^2} = \frac{t^2}{(t^2+1)^2},\ \sqrt{x(1-x)} = \frac{t}{t^2+1},\ t \in (0\,;\infty) \end{array}\right]$$
$$= \int \frac{t^2+1}{t}\frac{2t\,\mathrm{d}t}{(t^2+1)^2} = \int \frac{2\,\mathrm{d}t}{t^2+1} = 2\operatorname{arctg} t + c = 2\operatorname{arctg}\sqrt{\frac{x}{1-x}} + c, \text{ for } x \in (0\,;1).$$

---

$$\int \frac{1+x}{\sqrt{1-x^2}}\,\mathrm{d}x = \left[\begin{array}{c} 1 - x^2 > 0 \\ x \in (-1\,;1) \end{array}\right] = \int \frac{1+x}{\sqrt{(1-x)(1+x)}}\,\mathrm{d}x = \int \sqrt{\frac{1+x}{1-x}}\,\mathrm{d}x$$
$$= \left[\begin{array}{l} \sqrt{\frac{1+x}{1-x}} = t,\ \frac{1+x}{1-x} = t^2,\ 1 + x = t^2 - t^2x,\ x = \frac{t^1-1}{t^2+1} \\ \mathrm{d}x = \frac{2t(t^2+1)-(t^2-1)2t}{(t^2+1)^2}\,\mathrm{d}t = \frac{4t\,\mathrm{d}t}{(t^2+1)^2},\ t \in (0\,;\infty) \end{array}\right] = \int \frac{4t^2\,\mathrm{d}t}{(t^2+1)^2} = \int \frac{2t\cdot 2t\,\mathrm{d}t}{(t^2+1)^2}$$
$$= \left[\begin{array}{l|l} u = 2t & u' = 2 \\ v' = \frac{2t}{(t^2+1)^2} & v = -\frac{1}{t^2+1} \end{array}\right] = -\frac{2t}{t^2+1} - \int \frac{-2\,\mathrm{d}t}{t^2+1} = 2\int \frac{\mathrm{d}t}{t^2+1} - \frac{2t}{t^2+1}$$
$$= 2\operatorname{arctg} t - \frac{2t}{t^2+1} + c = 2\operatorname{arctg}\sqrt{\frac{1+x}{1-x}} - \frac{2\sqrt{\frac{1+x}{1-x}}}{\frac{1+x}{1-x}+1} + c = 2\operatorname{arctg}\sqrt{\frac{1+x}{1-x}} - \frac{2\sqrt{\frac{1+x}{1-x}}}{\frac{2}{1-x}} + c$$
$$= 2\operatorname{arctg}\sqrt{\frac{1+x}{1-x}} - \sqrt{1-x^2} + c = 2\operatorname{arctg}\sqrt{\frac{1+x}{1-x}} - \sqrt{1-x^2} + c, \text{ for } x \in (-1\,;1).$$

---

$$\int \frac{1+x}{\sqrt{1-x^2}}\,\mathrm{d}x = \int \frac{\mathrm{d}x}{\sqrt{1-x^2}} + \int \frac{x\,\mathrm{d}x}{\sqrt{1-x^2}} = \arcsin x - \frac{1}{2}\int \frac{\mathrm{d}t}{\sqrt{t}} = \left[\begin{array}{c} t = 1 - x^2 \\ \mathrm{d}t = -2x\,\mathrm{d}x,\ x\,\mathrm{d}x = -\frac{\mathrm{d}t}{2} \end{array}\right]$$
$$= \arcsin x - \frac{1}{2}\int t^{-\frac{1}{2}}\,\mathrm{d}t = \arcsin x - \frac{1}{2}\frac{t^{\frac{1}{2}}}{\frac{1}{2}} + c = \arcsin x - \sqrt{t} + c$$
$$= \arcsin x - \sqrt{1-x^2} + c, \text{ for } x \in (0\,;1).$$

---

$$\int \frac{dx}{\left(x-\sqrt{x^2-1}\right)^2} = \int \frac{1}{\left(x-\sqrt{x^2-1}\right)^2} \frac{\left(x+\sqrt{x^2-1}\right)^2}{\left(x+\sqrt{x^2-1}\right)^2} \, dx = \int \frac{x^2+2x\sqrt{x^2-1}+[x^2-1]}{(x^2-[x^2-1])^2} \, dx$$
$$= \int \frac{2x^2-1+2x\sqrt{x^2-1}}{1^2} \, dx = \int (2x^2-1) \, dx + \int 2x\sqrt{x^2-1} \, dx = \left[ \begin{array}{c} x^2-1=t \\ 2x\,dx = dt \end{array} \right]$$
$$= \int (2x^2-1) \, dx + \int t^{\frac{1}{2}} \, dt = \tfrac{2}{3}x^3 - x + \tfrac{2}{3}t^{\frac{3}{2}} + c = \tfrac{2}{3}x^3 - x + \tfrac{2}{3}\left(x^2-1\right)^{\frac{3}{2}} + c$$
$$= \tfrac{2}{3}x^3 - x + \tfrac{2}{3}\sqrt{(x^2-1)^3} + c, \text{ for } x \in R-(-1\,;\,1).$$

---

$$\int \frac{x^n}{x-1} \, dx = \int \frac{x^n-1+1}{x-1} \, dx = \int \frac{x^n-1}{x-1} \, dx + \int \frac{dx}{x-1} = \int \left[x^{n-1} + \cdots + x + 1\right] dx + \int \frac{dx}{x-1}$$
$$= \frac{x^n}{n} + \cdots + \frac{x^2}{2} + x + \ln|x-1| + c = \ln|x-1| + \sum_{k=1}^{n} \frac{x^k}{k} + c, \text{ for } x \in R-\{1\},\ n \in N.$$

---

$$\int \frac{x}{x-1} \, dx = \int \frac{x-1+1}{x-1} \, dx = \int dx + \int \frac{dx}{x-1} = x + \ln|x-1| + c, \text{ for } x \in R-\{1\}.$$

---

$$\int \frac{x^3}{x-1} \, dx = \int \frac{x^3-1+1}{x-1} \, dx = \int \left[x^2+x+1\right] dx + \int \frac{dx}{x-1} = \frac{x^3}{3} + \frac{x^2}{2} + x + \ln|x-1| + c,$$
for $x \in R-\{1\}$.

---

$$\int \frac{x^9}{x-1} \, dx = \int \frac{x^9-1+1}{x-1} \, dx = x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1+\ln|x-1|+c$$
$$= \frac{x^9}{9} + \frac{x^8}{8} + \frac{x^7}{7} + \frac{x^6}{6} + \frac{x^5}{5} + \frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{2} + x + \ln|x-1| + c, \text{ for } x \in R-\{1\}.$$

---

$$\int \frac{dx}{\cos x} = \left[ \begin{array}{cc} \boxed{\text{UTS}} & t = \operatorname{tg}\frac{x}{2} \\ dx = \frac{2\,dt}{1+t^2}, & \cos x = \frac{1-t^2}{1+t^2} \end{array} \right] = \int \frac{1+t^2}{1-t^2} \frac{2\,dt}{1+t^2} = \int \frac{2\,dt}{1-t^2} = -\int \frac{2\,dt}{t^2-1} = -\frac{2}{2}\ln\left|\frac{t-1}{t+1}\right| + c$$
$$= -\ln\left|\frac{t-1}{t+1}\right| + c = \ln\left|\frac{t+1}{t-1}\right| + c = \ln\left|\frac{\operatorname{tg}\frac{x}{2}+1}{\operatorname{tg}\frac{x}{2}-1}\right| + c, \text{ for } x \in R-\left\{\tfrac{\pi}{2}+k\pi\,;\,k \in Z\right\}.$$

---

$$\int \frac{dx}{\cos x} = \int \frac{\cos x\,dx}{\cos^2 x} = \int \frac{\cos x\,dx}{1-\sin^2 x} = \left[ \begin{array}{c} t = \sin x \\ dt = \cos x\,dx \end{array} \right] = \int \frac{dt}{1-t^2} = -\int \frac{dt}{t^2-1} = -\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + c$$
$$= \frac{1}{2}\ln\left|\frac{t+1}{t-1}\right| + c = \frac{1}{2}\ln\left|\frac{\sin x+1}{\sin x-1}\right| + c = \frac{1}{2}\ln\frac{1+\sin x}{1-\sin x} + c, \text{ for } x \in R-\left\{\tfrac{\pi}{2}+k\pi\,;\,k \in Z\right\}.$$

---

$$\int \frac{dx}{1+\cos x} = \left[ \begin{array}{c} \boxed{\text{UTS}}\ \ t = \operatorname{tg}\frac{x}{2},\ dx = \frac{2\,dt}{1+t^2} \\ 1+\cos x = 1 + \frac{1-t^2}{1+t^2} = \frac{2}{1+t^2} \end{array} \right] = \int \frac{1+t^2}{2} \frac{2\,dt}{1+t^2} = \int dt = t + c = \operatorname{tg}\frac{x}{2} + c,$$
for $x \in R-\{\pi+2k\pi\,;\,k \in Z\}$.

---

$$\int \frac{dx}{1+\cos x} = \int \frac{1-\cos x}{(1+\cos x)(1-\cos x)} \, dx = \int \frac{1-\cos x}{1-\cos^2 x} \, dx = \int \frac{dx}{\sin^2 x} - \int \frac{\cos x}{\sin^2 x} \, dx = \left[ \begin{array}{c} \sin x = t \\ \cos x\,dx = dt \end{array} \right]$$
$$= -\operatorname{cotg} x - \int \frac{dt}{t^2} = -\operatorname{cotg} x - \frac{t^{-1}}{-1} + c = -\operatorname{cotg} x + \frac{1}{t} + c = -\operatorname{cotg} x + \frac{1}{\sin x} + c$$
$$= -\frac{\cos x}{\sin x} + \frac{1}{\sin x} + c = \frac{1-\cos x}{\sin x} + c, \text{ for } x \in R-\{k\pi\,;\,k \in Z\}.$$

---

$$\int\frac{dx}{\sin x}=\left[\begin{array}{ll}\boxed{\text{UTS}} & t=\operatorname{tg}\frac{x}{2}\\ dx=\frac{2\,dt}{1+t^2}, & \sin x=\frac{2t}{1+t^2}\end{array}\right]=\int\frac{1+t^2}{2t}\,\frac{2\,dt}{1+t^2}=\int\frac{dt}{t}=\ln|t|+c=\ln\left|\operatorname{tg}\frac{x}{2}\right|+c,$$
for $x\in R-\{k\pi\,;\,k\in Z\}$.

---

$$\int\frac{dx}{\sin x}=\int\frac{\sin x\,dx}{\sin^2 x}=\int\frac{\sin x\,dx}{1-\cos^2 x}=\left[\begin{array}{c}t=\cos x\\ dt=-\sin x\,dx\end{array}\right]=-\int\frac{dt}{1-t^2}=\int\frac{dt}{t^2-1}=\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|+c$$
$$=\frac{1}{2}\ln\left|\frac{\cos x-1}{\cos x+1}\right|+c=\frac{1}{2}\ln\frac{1-\cos x}{1+\cos x}+c=\ln\sqrt{\frac{1-\cos x}{1+\cos x}}+c,\text{ for } x\in R-\{k\pi\,;\,k\in Z\}.$$

---

$$\int\frac{dx}{1+\sin x}=\left[\begin{array}{ll}\boxed{\text{UTS}} & t=\operatorname{tg}\frac{x}{2},\ dx=\frac{2\,dt}{1+t^2}\\ \multicolumn{2}{l}{1+\sin x=1+\frac{2t}{1+t^2}=\frac{1+2t+t^2}{1+t^2}}\end{array}\right]=\int\frac{1+t^2}{(1+t)^2}\,\frac{2\,dt}{1+t^2}=\int\frac{2\,dt}{(1+t)^2}=\left[\begin{array}{c}1+t=u\\ dt=du\end{array}\right]$$
$$=2\int\frac{du}{u^2}=2\frac{u^{-1}}{-1}+c=c-\frac{2}{u}=c-\frac{2}{t+1}=c-\frac{2}{\operatorname{tg}\frac{x}{2}+1},\text{ for } x\in R-\left\{-\frac{\pi}{2}+2k\pi\,;\,k\in Z\right\}.$$

---

$$\int\frac{dx}{1+\sin x}=\int\frac{1-\sin x}{(1+\sin x)(1-\sin x)}\,dx=\int\frac{1-\sin x}{1-\sin^2 x}\,dx=\int\frac{1-\sin x}{\cos^2 x}\,dx=\int\frac{dx}{\cos^2 x}-\int\frac{\sin x}{\cos^2 x}\,dx$$
$$=\left[\begin{array}{c}\cos x=t\\ -\sin x\,dx=dt\end{array}\right]=\operatorname{tg}x+\int\frac{dt}{t^2}=\operatorname{tg}x+\frac{t^{-1}}{-1}+c=\operatorname{tg}x-\frac{1}{t}+c=\operatorname{tg}x-\frac{1}{\cos x}+c$$
$$=\frac{\sin x}{\cos x}-\frac{1}{\cos x}+c=\frac{\sin x-1}{\cos x}+c,\text{ for } x\in R-\left\{\frac{\pi}{2}+k\pi\,;\,k\in Z\right\}.$$

---

$$\int\frac{\sin x\,dx}{\sqrt{\cos^5 x}}=\left[\begin{array}{c}\cos x=t\\ -\sin x\,dx=dt\end{array}\right]=\int\frac{-dt}{\sqrt{t^5}}=-\int t^{-\frac{5}{2}}\,dt=-\frac{t^{-\frac{3}{2}}}{-\frac{3}{2}}+c=\frac{2}{3}\frac{1}{\sqrt{t^3}}+c=\frac{2}{3}\frac{1}{\sqrt{\cos^3 x}}+c,$$
for $x\ \left(-\frac{\pi}{2}+2k\pi\,;\,\frac{\pi}{2}+2k\pi\right)$, $k\in Z$.

---

$$\int\frac{\cos x\,dx}{\sqrt[3]{\sin^2 x}}=\left[\begin{array}{c}\sin x=t\\ \cos x\,dx=dt\end{array}\right]=\int\frac{dt}{\sqrt[3]{t^2}}=\int t^{-\frac{2}{3}}\,dt=\frac{t^{\frac{1}{3}}}{\frac{1}{3}}+c=3\sqrt[3]{t}+c=3\sqrt[3]{\sin x}+c,$$
for $x\in(0+2k\pi\,;\,\pi+2k\pi)$, $k\in Z$.

---

$$\int\sin^2 x\,dx=\int\frac{1-\cos 2x}{2}\,dx=\frac{x}{2}-\frac{\sin 2x}{4}+c,\text{ for } x\in R.$$

---

$$\int\cos^2 x\,dx=\int\frac{1+\cos 2x}{2}\,dx=\frac{x}{2}+\frac{\sin 2x}{4}+c,\text{ for } x\in R.$$

---

$$I=\int\sin^2 x\,dx=\int\sin x\sin x\,dx=\left[\begin{array}{l|l}u=\sin x & u'=\cos x\\ v'=\sin x & v=-\cos x\end{array}\right]=-\cos x\sin x+\int\cos^2 x\,dx$$
$$=-\frac{\sin 2x}{2}+\int(1-\sin^2 x)\,dx=-\frac{\sin 2x}{2}+x-\int\sin^2 x\,dx$$
$$=\left[\boxed{\text{Equation}}\quad I=-\frac{\sin 2x}{2}+x-I\ \Rightarrow\ 2I=x-\frac{\sin 2x}{2}\right]=\frac{x}{2}-\frac{\sin 2x}{4}+c,\text{ for } x\in R.$$

---

$$\int \sin^2 x\,dx = \left[\begin{array}{l|l} u' = 1 & u = x \\ v = \sin^2 x & v' = 2\sin x\cos x = \sin 2x \end{array}\right] = x\sin^2 x - \int x\sin 2x\,dx$$
$$= \left[\begin{array}{l|l} u' = \sin 2x & u = -\frac{\cos 2x}{2} \\ v = x & v' = 1 \end{array}\right] = x\sin^2 x - \left[-\frac{x\cos 2x}{2} + \int \frac{\cos 2x}{2}\,dx\right]$$
$$= x\sin^2 x + \frac{x\cos 2x}{2} - \int \frac{\cos 2x}{2}\,dx = x\sin^2 x + \frac{x\cos 2x}{2} - \frac{\sin 2x}{4} + c, \text{ for } x \in R.$$

---

$$I = \int \cos^2 x\,dx = \int \cos x\cos x\,dx = \left[\begin{array}{l|l} u = \cos x & u' = -\sin x \\ v' = \cos x & v = \sin x \end{array}\right] = \sin x\cos x + \int \sin^2 x\,dx$$
$$= \frac{\sin 2x}{2} + \int (1 - \cos^2 x)\,dx = \frac{\sin 2x}{2} + x - \int \cos^2 x\,dx$$
$$= \left[\boxed{\text{Equation}}\quad I = \frac{\sin 2x}{2} + x - I \;\Rightarrow\; 2I = x + \frac{\sin 2x}{2}\right] = \frac{x}{2} + \frac{\sin 2x}{4} + c, \text{ for } x \in R.$$

---

$$\int \cos^2 x\,dx = \left[\begin{array}{l|l} u' = 1 & u = x \\ v = \cos^2 x & v' = -2\cos x\sin x = -\sin 2x \end{array}\right] = x\cos^2 x + \int x\sin 2x\,dx$$
$$= \left[\begin{array}{l|l} u' = \sin 2x & u = -\frac{\cos 2x}{2} \\ v = x & v' = 1 \end{array}\right] = x\cos^2 x + \left[-\frac{x\cos 2x}{2} + \int \frac{\cos 2x}{2}\,dx\right]$$
$$= x\cos^2 x - \frac{x\cos 2x}{2} + \int \frac{\cos 2x}{2}\,dx = x\cos^2 x - \frac{x\cos 2x}{2} + \frac{\sin 2x}{4} + c, \text{ for } x \in R.$$

---

$$\int \sin^3 x\,dx = \int (1 - \cos^2 x)\sin x\,dx = \left[\begin{array}{c} \cos x = t \\ -\sin x\,dx = dt \end{array}\right] = -\int (1 - t^2)\,dt = \int (t^2 - 1)\,dt$$
$$= \frac{t^3}{3} - t + c = \frac{\cos^3 x}{3} - \cos x + c, \text{ for } x \in R.$$

---

$$\int \cos^3 x\,dx = \int (1 - \sin^2 x)\sin x\,dx = \left[\begin{array}{c} \sin x = t \\ \cos x\,dx = dt \end{array}\right] = \int (1 - t^2)\,dt = t - \frac{t^3}{3} + c$$
$$= \sin x - \frac{\sin^3 x}{3} + c, \text{ for } x \in R.$$

---

$$I_n = \int \sin^n x\,dx = \int \sin x\sin^{n-1} x\,dx = \left[\begin{array}{l|l} u = \sin^{n-1} x & u' = (n-1)\sin^{n-2} x\cos x \\ v' = \sin x & v = -\cos x \end{array}\right]$$
$$= -\cos x\sin^{n-1} x + (n-1)\int \sin^{n-2} x\cos^2 x\,dx$$
$$= -\cos x\sin^{n-1} x + (n-1)\int \sin^{n-2} x(1 - \sin^2 x)\,dx$$
$$= -\cos x\sin^{n-1} x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx$$
$$= \left[\begin{array}{l} \boxed{\text{Equation}}\quad I_n = -\cos x\sin^{n-1} x + (n-1)I_{n-2} - (n-1)I_n \\ \qquad \Rightarrow\; I_n + (n-1)I_n = -\cos x\sin^{n-1} x + (n-1)I_{n-2} \;\Rightarrow\; I_n = -\frac{\cos x\sin^{n-1} x}{n} + \frac{n-1}{n}I_{n-2} \end{array}\right]$$
$$= -\frac{\cos x\sin^{n-1} x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx, \text{ for } x \in R,\; n = 3, 4, 5, \ldots.$$

---

$$\int \sin^{2n+1} x\,dx = \int \sin x\ \sin^{2n} x\,dx = \int \sin x\,(\sin^2 x)^n\,dx = \int \sin x\,(1-\cos^2 x)^n\,dx$$
$$= \left[\begin{array}{c}\cos x = t\\ -\sin x\,dx = dt\end{array}\right] = -\int (1-t^2)^n\,dt = -\int \Big[\sum_{j=0}^{n}\binom{n}{j}(-t^2)^j\Big]\,dt = -\int \Big[\sum_{j=0}^{n}\binom{n}{j}(-1)^j t^{2j}\Big]\,dt$$
$$= -\sum_{j=0}^{n}\binom{n}{j}(-1)^j\int t^{2j}\,dt = \sum_{j=0}^{n}\binom{n}{j}(-1)^{j+1}\frac{t^{2j+1}}{2j+1} + c = \sum_{j=0}^{n}\binom{n}{j}(-1)^{j+1}\frac{\cos^{2j+1} x}{2j+1} + c$$
$$= -\binom{n}{0}\cos x + \binom{n}{1}\frac{\cos^3 x}{3} - \binom{n}{2}\frac{\cos^5 x}{5} + \cdots + (-1)^{n+1}\binom{n}{n}\frac{\cos^{2n+1} x}{2n+1} + c, \text{ for } n\in N,\ x\in R.$$

---

$$I_n = \int \cos^n x\,dx = \int \cos x \cos^{n-1} x\,dx = \left[\begin{array}{l|l} u = \cos^{n-1} x & u' = -(n-1)\cos^{n-2} x \sin x\\ v' = \cos x & v = \sin x\end{array}\right]$$
$$= \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x \sin^2 x\,dx$$
$$= \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x(1-\cos^2 x)\,dx =$$
$$= \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx$$
$$= \left[\begin{array}{l}\boxed{\text{Equation}}\quad I_n = \sin x\cos^{n-1} x + (n-1)I_{n-2} - (n-1)I_n\\ \quad\Rightarrow\ I_n + (n-1)I_n = \sin x\cos^{n-1} x + (n-1)I_{n-2}\ \Rightarrow\ I_n = \frac{\sin x\cos^{n-1} x}{n} + \frac{n-1}{n}I_{n-2}\end{array}\right]$$
$$= \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx, \text{ for } x\in R,\ n = 3, 4, 5, \ldots.$$

---

$$\int \cos^{2n+1} x\,dx = \int \cos x\ \cos^{2n} x\,dx = \int \cos x\,(\cos^2 x)^n\,dx = \int \cos x\,(1-\sin^2 x)^n\,dx$$
$$= \left[\begin{array}{c}\sin x = t\\ \cos x\,dx = dt\end{array}\right] = \int (1-t^2)^n\,dt = \int \Big[\sum_{j=0}^{n}\binom{n}{j}(-t^2)^j\Big]\,dt = \int \Big[\sum_{j=0}^{n}\binom{n}{j}(-1)^j t^{2j}\Big]\,dt$$
$$= \sum_{j=0}^{n}\binom{n}{j}(-1)^j\int t^{2j}\,dt = \sum_{j=0}^{n}\binom{n}{j}(-1)^j\frac{t^{2j+1}}{2j+1} + c = \sum_{j=0}^{n}\binom{n}{j}(-1)^j\frac{\sin^{2j+1} x}{2j+1} + c$$
$$= \binom{n}{0}\sin x - \binom{n}{1}\frac{\sin^3 x}{3} + \binom{n}{2}\frac{\sin^5 x}{5} - \cdots + (-1)^n\binom{n}{n}\frac{\sin^{2n+1} x}{2n+1} + c, \text{ for } n\in N,\ x\in R.$$

---

$$\int \operatorname{tg}^2 x\,dx = \int \frac{\sin^2 x}{\cos^2 x}\,dx = \int \frac{1-\cos^2 x}{\cos^2 x}\,dx = \int \Big[\frac{1}{\cos^2 x} - 1\Big]\,dx = \operatorname{tg} x - x + c,$$
for $x\in R-\{\frac{\pi}{2} + k\pi\,;\ k\in Z\}$.

---

$$\int \operatorname{cotg}^2 x\,dx = \int \frac{\cos^2 x}{\sin^2 x}\,dx = \int \frac{1-\sin^2 x}{\sin^2 x}\,dx = \int \Big[\frac{1}{\sin^2 x} - 1\Big]\,dx = -\operatorname{cotg} x - x + c,$$
for $x\in R-\{k\pi\,;\ k\in Z\}$.

---

$$\int \frac{(\sin x - \cos x)\,dx}{\sqrt[4]{\sin x + \cos x}} = \left[\begin{array}{l}\boxed{\text{UTS}}\ \ t = \operatorname{tg}\frac{x}{2},\ \ dx = \frac{2\,dt}{1+t^2}\\ \cos x = \frac{1-t^2}{1+t^2},\ \ \sin x = \frac{2t}{1+t^2}\end{array}\right] = \int \frac{\Big(\frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2}\Big)\frac{2\,dt}{1+t^2}}{\sqrt[4]{\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}}} = \int \frac{(t^2+2t-1)\,dt}{(1+t^2)^2\ \sqrt[4]{\frac{1+2t-t^2}{1+t^2}}}$$
$$= \cdots ☹$$

---

$$\int \frac{(\sin x - \cos x)\,\mathrm{d}x}{\sqrt[4]{\sin x + \cos x}} = \left[\begin{array}{c} \sin x + \cos x = t \\ (\cos x - \sin x)\,\mathrm{d}x = \mathrm{d}t \end{array}\right] = \int \frac{-\mathrm{d}t}{\sqrt[4]{t}} = -\int t^{-\frac{1}{4}}\,\mathrm{d}t = -\frac{t^{\frac{3}{4}}}{\frac{3}{4}} + c$$
$$= c - \frac{4}{3}\sqrt[4]{(\sin x + \cos x)^3}, \text{ for } x \in \left(\frac{\pi}{2} + 2k\pi\,;\ \frac{3\pi}{2} + 2k\pi\right),\ k \in Z.$$

---

$$\int \frac{x^2}{\sin x^3}\,\mathrm{d}x = \left[\begin{array}{c} t = x^3 \\ \mathrm{d}t = 3x^2\,\mathrm{d}x \end{array}\right] = \frac{1}{3}\int \frac{\mathrm{d}t}{\sin t} = \left[\begin{array}{cc} \boxed{\text{UTS}} & u = \operatorname{tg}\frac{t}{2} \\ \mathrm{d}t = \frac{2\,\mathrm{d}u}{1+u^2}, & \sin t = \frac{2u}{1+u^2} \end{array}\right] = \frac{1}{3}\int \frac{1+u^2}{2u}\,\frac{2\,\mathrm{d}u}{1+u^2}$$
$$= \frac{1}{3}\int \frac{\mathrm{d}u}{u} = \frac{1}{3}\ln|u| + c = \frac{1}{3}\ln\left|\operatorname{tg}\frac{x^3}{2}\right| + c, \text{ for } x \in R - \{\sqrt[3]{k\pi}\,;\ k \in Z\}.$$

---

$$\int \frac{x^2}{\sin x^3}\,\mathrm{d}x = \left[\begin{array}{c} t = x^3 \\ \mathrm{d}t = 3x^2\,\mathrm{d}x \end{array}\right] = \frac{1}{3}\int \frac{\mathrm{d}t}{\sin t} = \frac{1}{3}\int \frac{\sin t\,\mathrm{d}t}{\sin^2 t} = \frac{1}{3}\int \frac{\sin t\,\mathrm{d}t}{1-\cos^2 t} = \left[\begin{array}{c} u = \cos t = \cos x^3 \\ \mathrm{d}u = -\sin t\,\mathrm{d}t \end{array}\right]$$
$$= \frac{1}{3}\int \frac{-\mathrm{d}u}{1-u^2} = \frac{1}{3}\int \frac{\mathrm{d}u}{u^2-1} = \frac{1}{3}\frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + c = \frac{1}{6}\ln\left|\frac{u-1}{u+1}\right| + c = \frac{1}{6}\ln\left|\frac{\cos x^3 - 1}{\cos x^3 + 1}\right| + c,$$
for $x \in R - \{\sqrt[3]{k\pi}\,;\ k \in Z\}$.

---

$$\int \frac{\mathrm{d}x}{\sin^2 x \cos^2 x} = \int \frac{4\,\mathrm{d}x}{(2\sin x \cos x)^2} = \int \frac{4\,\mathrm{d}x}{\sin^2 2x} = \left[\begin{array}{c} 2x = t \\ 2\,\mathrm{d}x = \mathrm{d}t \end{array}\right] = \int \frac{2\,\mathrm{d}t}{\sin^2 t} = -2\operatorname{cotg} t + c$$
$$= -2\operatorname{cotg} 2x + c, \text{ for } x \in R - \{\tfrac{k\pi}{2}\,;\ k \in Z\}.$$

---

$$\int \frac{\mathrm{d}x}{\sin^2 x \cos^2 x} = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x}\,\mathrm{d}x = \int \frac{\mathrm{d}x}{\sin^2 x} + \int \frac{\mathrm{d}x}{\cos^2 x} = \operatorname{tg} x - \operatorname{cotg} x + c = \frac{\sin x}{\cos x} - \frac{\cos x}{\sin x} + c$$
$$= 2\frac{\sin^2 x - \cos^2 x}{2\sin x \cos x} + c = 2\frac{-\cos 2x}{\sin 2x} + c = -2\operatorname{cotg} 2x + c, \text{ for } x \in R - \{\tfrac{k\pi}{2}\,;\ k \in Z\}.$$

---

$$I = \int \cos ax \sin bx\,\mathrm{d}x = \left[\begin{array}{l|l} u = \sin bx & u' = b\cos bx \\ v' = \cos ax & v = \frac{1}{a}\sin ax \end{array}\right] = \frac{\sin ax \sin bx}{a} - \frac{b}{a}\int \sin ax \cos bx\,\mathrm{d}x$$
$$= \left[\begin{array}{l|l} u = \cos bx & u' = -b\sin bx \\ v' = \sin ax & v = -\frac{1}{a}\cos ax \end{array}\right] = \frac{\sin ax \sin bx}{a} - \frac{b}{a}\left[-\frac{\cos ax \cos bx}{a} - \frac{b}{a}\int \cos ax \sin bx\,\mathrm{d}x\right]$$
$$= \frac{1}{a}\sin ax \sin bx + \frac{b}{a^2}\cos ax \cos bx + \frac{b^2}{a^2}\int \cos ax \sin bx\,\mathrm{d}x$$
$$= \left[\begin{array}{l} \boxed{\text{Equation}} \quad I = \frac{1}{a}\sin ax \sin bx + \frac{b}{a^2}\cos ax \cos bx + \frac{b^2}{a^2} I \\ \qquad \Rightarrow \left(1 - \frac{b^2}{a^2}\right) I = \frac{1}{a}\sin ax \sin bx + \frac{b}{a^2}\cos ax \cos bx,\ 1 - \frac{b^2}{a^2} = \frac{a^2-b^2}{a^2} \end{array}\right]$$
$$= \frac{a}{a^2-b^2}\sin ax \sin bx + \frac{b}{a^2-b^2}\cos ax \cos bx + c, \text{ for } a, b \in R - \{0\},\ a \neq b,\ x \in R.$$

---

$$I = \int \cos ax \sin bx\,\mathrm{d}x = \left[\begin{array}{l|l} u = \cos ax & u' = -a\sin ax \\ v' = \sin bx & v = -\frac{1}{b}\cos bx \end{array}\right] = -\frac{\cos ax \cos bx}{b} - \frac{a}{b}\int \sin ax \cos bx\,\mathrm{d}x$$
$$= \left[\begin{array}{l|l} u = \sin ax & u' = a\cos ax \\ v' = \cos bx & v = \frac{1}{b}\sin bx \end{array}\right] = -\frac{\cos ax \cos bx}{b} - \frac{a}{b}\left[\frac{\sin ax \sin bx}{b} - \frac{a}{b}\int \cos ax \sin bx\,\mathrm{d}x\right]$$
$$= -\frac{1}{b}\cos ax \cos bx - \frac{a}{b^2}\sin ax \sin bx + \frac{a^2}{b^2}\int \cos ax \sin bx\,\mathrm{d}x$$
$$= \left[\begin{array}{l} \boxed{\text{Equation}} \quad I = -\frac{1}{b}\cos ax \cos bx - \frac{a}{b^2}\sin ax \sin bx + \frac{a^2}{b^2} I \\ \qquad \Rightarrow \left(1 - \frac{a^2}{b^2}\right) I = -\frac{1}{b}\cos ax \cos bx - \frac{a}{b^2}\sin ax \sin bx,\ 1 - \frac{a^2}{b^2} = \frac{b^2-a^2}{b^2} \end{array}\right]$$
$$= -\frac{b}{b^2-a^2}\cos ax \cos bx - \frac{a}{b^2-a^2}\sin ax \sin bx + c, \text{ for } a, b \in R - \{0\},\ a \neq b,\ x \in R.$$

---

$$\int \cos ax \sin bx \, dx = \left[ \sin\alpha\cos\beta = \tfrac{\sin(\alpha-\beta)+\sin(\alpha+\beta)}{2} \right] = \int \tfrac{\sin(bx-ax)+\sin(bx+ax)}{2}\, dx$$
$$= \int \left[ \tfrac{\sin(b-a)x}{2} + \tfrac{\sin(b+a)x}{2} \right] dx = -\tfrac{\cos(b-a)x}{2(b-a)} - \tfrac{\cos(b+a)x}{2(b+a)} + c$$
$$= -\tfrac{(b+a)\cos(bx-ax)+(b-a)\cos(bx+ax)}{2(b-a)(b+a)} + c = \left[ \cos(\alpha\pm\beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \right]$$
$$= \tfrac{(b+a)[\cos bx\cos ax+\sin bx\sin ax]+(b-a)[\cos bx\cos ax-\sin bx\sin ax]}{2(a-b)(a+b)} + c$$
$$= \tfrac{2b\cos bx\cos ax+2a\sin bx\sin ax}{2(a^2-b^2)} + c = \tfrac{b\cos ax\cos bx+a\sin ax\sin bx}{a^2-b^2} + c,$$
for $a, b \in R-\{0\},\ a \neq b,\ x \in R$.

---

$$\int \cos ax \sin ax \, dx = \left[ 2\sin\alpha\cos\alpha = \sin 2\alpha \right] = \int \tfrac{\sin 2ax}{2}\, dx = \tfrac{-\cos 2ax}{2\cdot 2a} + c = c - \tfrac{\cos 2ax}{4a},$$
for $a \in R-\{0\},\ x \in R$.

---

$$I = \int \cos ax \cos bx \, dx = \left[ \begin{array}{l|l} u = \cos bx & u' = -b\sin bx \\ v' = \cos ax & v = \frac{1}{a}\sin ax \end{array} \right] = \tfrac{\sin ax\cos bx}{a} + \tfrac{b}{a}\int \sin ax \sin bx \, dx$$
$$= \left[ \begin{array}{l|l} u = \sin bx & u' = b\cos bx \\ v' = \sin ax & v = -\frac{1}{a}\cos ax \end{array} \right] = \tfrac{\sin ax\cos bx}{a} + \tfrac{b}{a}\left[ -\tfrac{\cos ax\sin bx}{a} + \tfrac{b}{a}\int \cos ax\cos bx\, dx \right]$$
$$= \tfrac{1}{a}\sin ax\cos bx - \tfrac{b}{a^2}\cos ax\sin bx + \tfrac{b^2}{a^2}\int \cos ax\cos bx\, dx =$$
$$\left[ \boxed{\text{Equation}} \quad I = \tfrac{1}{a}\sin ax\cos bx - \tfrac{b}{a^2}\cos ax\sin bx + \tfrac{b^2}{a^2} I \quad \Rightarrow \quad \left(1-\tfrac{b^2}{a^2}\right) I = \tfrac{1}{a}\sin ax\cos bx - \tfrac{b}{a^2}\cos ax\sin bx + c_1, \quad 1-\tfrac{b^2}{a^2} = \tfrac{a^2-b^2}{a^2} \right]$$
$$= \tfrac{a}{a^2-b^2}\sin ax\cos bx - \tfrac{b}{a^2-b^2}\cos ax\sin bx + c,$$ for $a, b \in R-\{0\},\ a \neq b,\ x \in R$.

---

$$I = \int \cos ax \cos bx \, dx = \left[ \begin{array}{l|l} u = \cos ax & u' = -a\sin ax \\ v' = \cos bx & v = -\frac{1}{b}\sin bx \end{array} \right] = \tfrac{\cos ax\sin bx}{b} + \tfrac{a}{b}\int \sin ax \sin bx \, dx$$
$$= \left[ \begin{array}{l|l} u = \sin ax & u' = a\cos ax \\ v' = \sin bx & v = -\frac{1}{b}\cos bx \end{array} \right] = \tfrac{\cos ax\sin bx}{b} + \tfrac{a}{b}\left[ -\tfrac{\sin ax\cos bx}{b} + \tfrac{a}{b}\int \cos ax\cos bx\, dx \right]$$
$$= \tfrac{1}{b}\cos ax\sin bx - \tfrac{a}{b^2}\sin ax\cos bx + \tfrac{a^2}{b^2}\int \cos ax\cos bx\, dx$$
$$= \left[ \boxed{\text{Equation}} \quad I = \tfrac{1}{b}\cos ax\sin bx - \tfrac{a}{b^2}\sin ax\cos bx + \tfrac{a^2}{b^2} I \quad \Rightarrow \quad \left(1-\tfrac{a^2}{b^2}\right) I = \tfrac{1}{b}\cos ax\sin bx - \tfrac{a}{b^2}\sin ax\cos bx + c_1, \quad 1-\tfrac{a^2}{b^2} = \tfrac{b^2-a^2}{b^2} \right]$$
$$= \tfrac{b}{b^2-a^2}\cos ax\sin bx - \tfrac{a}{b^2-a^2}\sin ax\cos bx + c,$$ for $a, b \in R-\{0\},\ a \neq b,\ x \in R$.

---

$$\int \cos ax \cos bx \, dx = \left[ \cos\alpha\cos\beta = \tfrac{\cos(\alpha+\beta)+\cos(\alpha-\beta)}{2} \right] = \int \tfrac{\cos(ax+bx)+\cos(ax-bx)}{2}\, dx$$
$$= \int \left[ \tfrac{\cos(a+b)x}{2} + \tfrac{\cos(a-b)x}{2} \right] dx = \tfrac{\sin(a+b)x}{2(a+b)} + \tfrac{\sin(a-b)x}{2(a-b)} + c$$
$$= \tfrac{(a-b)\sin(ax+bx)+(a+b)\sin(ax-bx)}{2(a-b)(a+b)} + c = \left[ \sin(\alpha\pm\beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \right]$$
$$= \tfrac{(a-b)[\sin ax\cos bx+\cos ax\sin bx]+(a+b)[\sin ax\cos bx-\cos ax\sin bx]}{2(a-b)(a+b)} + c$$
$$= \tfrac{2a\sin ax\cos bx-2b\sin bx\cos ax}{2(a^2-b^2)} + c = \tfrac{a\sin ax\cos bx-b\cos ax\sin bx}{a^2-b^2} + c,$$
for $a, b \in R-\{0\},\ a \neq b,\ x \in R$.

---

$$\int \cos^2 ax\,\mathrm{d}x = \left[\ \cos^2\alpha = \tfrac{1+\cos 2\alpha}{2}\ \right] = \int \tfrac{1+\cos 2ax}{2}\,\mathrm{d}x = \int \left[\tfrac{1}{2} + \tfrac{\cos 2ax}{2}\right]\mathrm{d}x = \tfrac{x}{2} + \tfrac{\sin 2ax}{2\cdot 2a} + c$$
$$= \tfrac{x}{2} + \tfrac{\sin 2ax}{4a} + c, \text{ for } a\in R-\{0\},\ x\in R.$$

---

$$I = \int \sin ax \sin bx\,\mathrm{d}x = \left[\begin{array}{l|l} u = \sin bx & u' = b\cos bx \\ v' = \sin ax & v = -\frac{1}{a}\cos ax \end{array}\right] = -\tfrac{\cos ax \sin bx}{a} + \tfrac{b}{a}\int \cos ax \cos bx\,\mathrm{d}x$$
$$= \left[\begin{array}{l|l} u = \cos bx & u' = -b\sin bx \\ v' = \cos ax & v = \frac{1}{a}\sin ax \end{array}\right] = -\tfrac{\cos ax \sin bx}{a} + \tfrac{b}{a}\left[\tfrac{\sin ax \cos bx}{a} + \tfrac{b}{a}\int \sin ax \sin bx\,\mathrm{d}x\right]$$
$$= -\tfrac{1}{a}\cos ax \sin bx + \tfrac{b}{a^2}\sin ax \cos bx + \tfrac{b^2}{a^2}\int \sin ax \sin bx\,\mathrm{d}x =$$
$$\left[\ \boxed{\text{Equation}}\quad I = -\tfrac{1}{a}\cos ax \sin bx + \tfrac{b}{a^2}\sin ax \cos bx + \tfrac{b^2}{a^2} I \quad \Rightarrow\ \left(1-\tfrac{b^2}{a^2}\right)I = -\tfrac{1}{a}\cos ax \sin bx + \tfrac{b}{a^2}\sin ax \cos bx + c_1,\quad 1-\tfrac{b^2}{a^2} = \tfrac{a^2-b^2}{a^2}\ \right]$$
$$= -\tfrac{a}{a^2-b^2}\cos ax \sin bx + \tfrac{b}{a^2-b^2}\sin ax \cos bx + c, \text{ for } a,b\in R-\{0\},\ a\neq b,\ x\in R.$$

---

$$I = \int \sin ax \sin bx\,\mathrm{d}x = \left[\begin{array}{l|l} u = \sin ax & u' = a\cos ax \\ v' = \sin bx & v = -\frac{1}{b}\cos bx \end{array}\right] = -\tfrac{\sin ax \cos bx}{b} + \tfrac{a}{b}\int \cos ax \cos bx\,\mathrm{d}x$$
$$= \left[\begin{array}{l|l} u = \cos ax & u' = -a\sin ax \\ v' = \cos bx & v = \frac{1}{b}\sin bx \end{array}\right] = -\tfrac{\sin ax \cos bx}{b} + \tfrac{a}{b}\left[\tfrac{\cos ax \sin bx}{b} + \tfrac{a}{b}\int \sin ax \sin bx\,\mathrm{d}x\right]$$
$$= -\tfrac{1}{b}\sin ax \cos bx + \tfrac{a}{b^2}\cos ax \sin bx + \tfrac{a^2}{b^2}\int \sin ax \sin bx\,\mathrm{d}x =$$
$$\left[\ \boxed{\text{Equation}}\quad I = -\tfrac{1}{b}\sin ax \cos bx + \tfrac{a}{b^2}\cos ax \sin bx + \tfrac{a^2}{b^2} I \quad \Rightarrow\ \left(1-\tfrac{a^2}{b^2}\right)I = -\tfrac{1}{b}\sin ax \cos bx + \tfrac{a}{b^2}\cos ax \sin bx + c_1,\quad 1-\tfrac{a^2}{b^2} = \tfrac{b^2-a^2}{b^2}\ \right]$$
$$= -\tfrac{b}{b^2-a^2}\sin ax \cos bx + \tfrac{a}{b^2-a^2}\cos ax \sin bx + c, \text{ for } a,b\in R-\{0\},\ a\neq b,\ x\in R.$$

---

$$\int \sin ax \sin bx\,\mathrm{d}x = \left[\ \sin\alpha \sin\beta = \tfrac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2}\ \right] = \int \tfrac{\cos(ax-bx)-\cos(ax+bx)}{2}\,\mathrm{d}x$$
$$= \int \left[\tfrac{\cos(a-b)x}{2} - \tfrac{\cos(a+b)x}{2}\right]\mathrm{d}x = \tfrac{\sin(a-b)x}{2(a-b)} - \tfrac{\sin(a+b)x}{2(a+b)} + c$$
$$= \tfrac{(a+b)\sin(ax-bx) - (a-b)\sin(ax+bx)}{2(a-b)(a+b)} + c = \left[\ \sin(\alpha\pm\beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta\ \right]$$
$$= \tfrac{(a+b)[\sin ax \cos bx - \cos ax \sin bx] - (a-b)[\sin ax \cos bx + \cos ax \sin bx]}{2(a-b)(a+b)} + c$$
$$= \tfrac{2b\sin ax \cos bx - 2a\sin bx \cos ax}{2(a^2-b^2)} + c = \tfrac{b\sin ax \cos bx - a\cos ax \sin bx}{a^2-b^2} + c,$$
for $a,b\in R-\{0\},\ a\neq b,\ x\in R$.

---

$$\int \sin^2 ax\,\mathrm{d}x = \left[\ \sin^2\alpha = \tfrac{1-\cos 2\alpha}{2}\ \right] = \int \tfrac{1-\cos 2ax}{2}\,\mathrm{d}x = \int \left[\tfrac{1}{2} - \tfrac{\cos 2ax}{2}\right]\mathrm{d}x = \tfrac{x}{2} - \tfrac{\sin 2ax}{2\cdot 2a} + c$$
$$= \tfrac{x}{2} - \tfrac{\sin 2ax}{4a} + c, \text{ for } a\in R-\{0\},\ x\in R.$$

---

$$\int x^2 \ln x\,\mathrm{d}x = \left[\begin{array}{l|l} u = \ln x & u' = \frac{1}{x} \\ v' = x^2 & v = \frac{x^3}{3} \end{array}\right] = \tfrac{x^3}{3}\ln x - \int \tfrac{x^3}{3}\tfrac{1}{x}\,\mathrm{d}x = \tfrac{x^3}{3}\ln x - \int \tfrac{x^2}{3}\,\mathrm{d}x$$
$$= \tfrac{x^3}{3}\ln x - \tfrac{x^3}{9} + c, \text{ for } x > 0.$$

---

$$\int x\ln^2 x\,\mathrm{d}x = \left[\begin{array}{l|l} u=\ln^2 x & u'=\frac{2\ln x}{x} \\ v'=x & v=\frac{x^2}{2}\end{array}\right] = \tfrac{x^2}{2}\ln^2 x - \int \tfrac{x^2}{2}\tfrac{2\ln x}{x}\,\mathrm{d}x = \tfrac{x^2}{2}\ln^2 x - \int x\ln x\,\mathrm{d}x$$
$$= \left[\begin{array}{l|l} u=\ln x & u'=\frac{1}{x} \\ v'=x & v=\frac{x^2}{2}\end{array}\right] = \tfrac{x^2}{2}\ln^2 x - \left[\tfrac{x^2}{2}\ln x - \int \tfrac{x^2}{2}\tfrac{1}{x}\,\mathrm{d}x\right] = \tfrac{x^2}{2}\ln^2 x - \tfrac{x^2}{2}\ln x + \int \tfrac{x\,\mathrm{d}x}{2}$$
$$= \tfrac{x^2}{2}\ln^2 x - \tfrac{x^2}{2}\ln x + \tfrac{x^2}{4} + c, \text{ for } x>0.$$

---

$$\int \frac{\mathrm{d}x}{(1-x)x^2} = \left[\begin{array}{l} \frac{1}{(1-x)x^2} = \frac{A}{x-1} + \frac{B}{x^2} + \frac{C}{x} \\ A=1,\ B=-1,\ C=-1\end{array}\right] = \int \left[\tfrac{1}{x-1} - \tfrac{1}{x^2} - \tfrac{1}{x}\right]\mathrm{d}x$$
$$= \ln|x-1| - \tfrac{x^{-1}}{-1} - \ln|x| + c = \ln\left|\tfrac{x-1}{x}\right| + \tfrac{1}{x} + c, \text{ for } x\in R-\{0,1\}.$$

---

$$\int\sqrt{a^2-x^2}\,\mathrm{d}x = \left[\begin{array}{c} x=a\sin t,\ t=\arcsin\frac{x}{a},\ \mathrm{d}x=a\cos t\,\mathrm{d}t,\ x\in(-a\,;\,a),\ t\in\left(-\frac{\pi}{2}\,;\,\frac{\pi}{2}\right) \\ \sqrt{a^2-x^2}=\sqrt{a^2-a^2\sin^2 t}=a\sqrt{1-\sin^2 t}=a\sqrt{\cos^2 t}=a\,|\cos t|=a\cos t\end{array}\right]$$
$$= \int a^2\cos^2 t\,\mathrm{d}t = a^2\int \tfrac{1+\cos 2t}{2}\,\mathrm{d}t = \tfrac{a^2}{2}t + a^2\tfrac{\sin 2t}{4} + c = \tfrac{a^2}{2}t + a^2\tfrac{2\sin t\cos t}{4} + c$$
$$= \tfrac{a^2}{2}t + \tfrac{a\sin t\cdot a\cos t}{2} + c = \tfrac{a^2}{2}\arcsin\tfrac{x}{a} + \tfrac{x\sqrt{a^2-x^2}}{2} + c, \text{ for } a>0,\ x\in(-a\,;\,a).$$

---

$$I = \int\sqrt{a^2-x^2}\,\mathrm{d}x = \int\frac{a^2-x^2}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \int\frac{a^2\,\mathrm{d}x}{\sqrt{a^2-x^2}} - \int\frac{x^2\,\mathrm{d}x}{\sqrt{a^2-x^2}} = \int\frac{a^2\,\mathrm{d}x}{\sqrt{a^2-x^2}} - \int\frac{x\cdot x\,\mathrm{d}x}{\sqrt{a^2-x^2}}$$
$$= \left[\begin{array}{l|l} u=x & u'=1 \\ v'=\frac{x}{\sqrt{a^2-x^2}}=\frac{2x}{2\sqrt{a^2-x^2}} & v=-\sqrt{a^2-x^2}\end{array}\right]$$
$$= \int\frac{a^2\,\mathrm{d}x}{\sqrt{a^2-x^2}} - \left[-x\sqrt{a^2-x^2} + \int\sqrt{a^2-x^2}\,\mathrm{d}x\right] = a^2\arcsin\tfrac{x}{a} + x\sqrt{a^2-x^2} - \int\sqrt{a^2-x^2}\,\mathrm{d}x$$
$$= \left[\ \boxed{\text{Equation}}\quad I = a^2\arcsin\tfrac{x}{a} + x\sqrt{a^2-x^2} - I \ \Rightarrow\ 2I = a^2\arcsin\tfrac{x}{a} + x\sqrt{a^2-x^2}\ \right]$$
$$= \tfrac{a^2}{2}\arcsin\tfrac{x}{a} + \tfrac{x\sqrt{a^2-x^2}}{2} + c, \text{ for } a>0,\ x\in(-a\,;\,a).$$

---

$$\int\sqrt{a^2-x^2}\,\mathrm{d}x = \left[\begin{array}{c} \boxed{\text{2nd ES}}\ \sqrt{a^2-x^2}=a-xt,\ t=\frac{a-\sqrt{a^2-x^2}}{x},\ x\in(-a\,;\,a),x\neq 0,t\in(-1\,;\,1),t\neq 0 \\ a^2-x^2=x^2t^2-2axt+a^2,\ 2axt=x^2+x^2t^2,\ 2at=x+xt^2,\ x=\frac{2at}{1+t^2} \\ \mathrm{d}x=\frac{2a(1+t^2)-2at\cdot 2t}{(1+t^2)^2}\,\mathrm{d}t=\frac{2a(1-t^2)}{(1+t^2)^2}\,\mathrm{d}t,\ \sqrt{a^2-x^2}=a-xt=a-\frac{2at^2}{1+t^2}=\frac{a-at^2}{1+t^2}\end{array}\right]$$
$$= \int\frac{a(1-t^2)}{1+t^2}\frac{2a(1-t^2)}{(1+t^2)^2}\,\mathrm{d}t = 2a^2\int\frac{(1-t^2)^2}{(1+t^2)^3}\,\mathrm{d}t = 2a^2\int\frac{1-2t^2+t^4}{(1+t^2)^3}\,\mathrm{d}t = 2a^2\int\frac{1+2t^2+t^4-4t^2}{(1+t^2)^3}\,\mathrm{d}t$$
$$= 2a^2\int\frac{(1+t^2)^2-4t^2-4+4}{(1+t^2)^3}\,\mathrm{d}t = 2a^2\int\frac{(1+t^2)^2-4(1+t^2)+4}{(1+t^2)^3}\,\mathrm{d}t = 2a^2\int\left[\frac{1}{1+t^2} - \frac{4}{(1+t^2)^2} + \frac{4}{(1+t^2)^3}\right]\mathrm{d}t$$
$$= \left[\ \text{p. 21}: \int\frac{\mathrm{d}t}{(1+t^2)^2} = \tfrac{1}{2}\operatorname{arctg} t + \tfrac{t}{2(t^2+1)},\ t\in R,\quad \int\frac{\mathrm{d}t}{(1+t^2)^3} = \tfrac{3}{8}\operatorname{arctg} t + \tfrac{3t}{8(t^2+1)} + \tfrac{t}{4(t^2+1)^2},\ t\in R\ \right]$$
$$= 2a^2\left[\operatorname{arctg} t - 4\left(\tfrac{1}{2}\operatorname{arctg} t + \tfrac{1}{2}\tfrac{t}{1+t^2}\right) + 4\left(\tfrac{3}{8}\operatorname{arctg} t + \tfrac{3}{8}\tfrac{t}{1+t^2} + \tfrac{1}{4}\tfrac{t}{(1+t^2)^2}\right)\right] + c$$
$$= a^2\operatorname{arctg} t - \tfrac{a^2t}{1+t^2} + \tfrac{2a^2t}{(1+t^2)^2} + c = a^2\operatorname{arctg} t - \tfrac{a}{2}\tfrac{2at}{1+t^2} + \tfrac{1}{2t}\tfrac{(2at)^2}{(1+t^2)^2} + c$$
$$= \left[\ \tfrac{1}{t} = \tfrac{x}{a-\sqrt{a^2-x^2}} = \tfrac{x}{a-\sqrt{a^2-x^2}}\tfrac{a+\sqrt{a^2-x^2}}{a+\sqrt{a^2-x^2}} = \tfrac{x(a+\sqrt{a^2-x^2})}{a^2-(a^2-x^2)} = \tfrac{a+\sqrt{a^2-x^2}}{x}\ \right]$$
$$= a^2\operatorname{arctg}\tfrac{a-\sqrt{a^2-x^2}}{x} - \tfrac{a}{2}x + \tfrac{a+\sqrt{a^2-x^2}}{2x}x^2 + c = a^2\operatorname{arctg}\tfrac{a-\sqrt{a^2-x^2}}{x} + \tfrac{x\sqrt{a^2-x^2}}{2} + c,$$
for $a>0,\ x\in(-a\,;\,a)$.

---

$$\int\sqrt{a^2+x^2}\,\mathrm{d}x = \left[\begin{array}{c} x=a\sinh t,\quad \mathrm{d}x=a\cosh t\,\mathrm{d}t,\quad x\in R,\ t\in R \\ \sqrt{a^2+x^2}=\sqrt{a^2+a^2\sinh^2 t}=a\sqrt{1+\sinh^2 t}=a\sqrt{\cosh^2 t}=a\,|\cosh t|=a\cosh t \end{array}\right]$$
$$= \int a^2\cosh^2 t\,\mathrm{d}t = \left[\begin{array}{l|l} u\ =a\cosh t & u'=a\sinh t \\ v'=a\cosh t & v\ =a\sinh t \end{array}\right] = a^2\sinh t\cosh t - \int a^2\sinh^2 t\,\mathrm{d}t$$
$$= \left[\begin{array}{l|l} u\ =a\sinh t & u'=a\cosh t \\ v'=a\sinh t & v\ =a\cosh t \end{array}\right] = a^2\sinh t\cosh t - \left[a^2\sinh t\cosh t - \int a^2\cosh^2 t\right]$$
$$= \int a^2\cosh^2 t\,\mathrm{d}t = \cdots ☹$$

---

$$\int\sqrt{a^2+x^2}\,\mathrm{d}x = \left[\begin{array}{c} x=a\sinh t,\quad \mathrm{d}x=a\cosh t\,\mathrm{d}t,\quad x\in R,\ t\in R \\ \sqrt{a^2+x^2}=\sqrt{a^2+a^2\sinh^2 t}=a\sqrt{1+\sinh^2 t}=a\sqrt{\cosh^2 t}=a\,|\cosh t|=a\cosh t \end{array}\right]$$
$$= \int a^2\cosh^2 t\,\mathrm{d}t = a^2\int\left(\frac{\mathrm{e}^t+\mathrm{e}^{-t}}{2}\right)^2\mathrm{d}t = a^2\int\frac{\mathrm{e}^{2t}+2+\mathrm{e}^{-2t}}{4}\,\mathrm{d}t = a^2\int\left[\frac{\mathrm{e}^{2t}}{4}+\frac{1}{2}+\frac{\mathrm{e}^{-2t}}{4}\right]\mathrm{d}t$$
$$= a^2\left[\frac{\mathrm{e}^{2t}}{8}+\frac{t}{2}-\frac{\mathrm{e}^{-2t}}{8}\right]+c_1 = \frac{a^2}{4}\,\frac{\mathrm{e}^{2t}-\mathrm{e}^{-2t}}{2}+\frac{a^2t}{2}+c = \frac{a^2}{4}\sinh 2t+\frac{a^2t}{2}+c$$
$$= \left[\begin{array}{c} t=\operatorname{argsinh}\frac{x}{a}=\ln\left(\frac{x}{a}+\sqrt{\frac{x^2}{a^2}+1}\right)=\ln\frac{x+\sqrt{a^2+x^2}}{a}=\ln\left(x+\sqrt{a^2+x^2}\right)-\ln a \\ a^2\sinh 2t=2a\sinh t\cdot a\cosh t=2x\sqrt{a^2+x^2} \end{array}\right]$$
$$= \frac{x\sqrt{a^2+x^2}}{2}+\frac{a^2\left[\ln\left(x+\sqrt{a^2+x^2}\right)-\ln a\right]}{2}+c_1 = \frac{x\sqrt{a^2+x^2}}{2}+\frac{a^2}{2}\ln\left(x+\sqrt{a^2+x^2}\right)+c_1-\frac{a^2}{2}\ln a$$
$$= \left[\ c_1-\frac{a^2}{2}\ln a=c=\text{const.}\ \right] = \frac{x\sqrt{a^2+x^2}}{2}+\frac{a^2}{2}\ln\left(x+\sqrt{a^2+x^2}\right)+c,\ \text{for } a>0,\ x\in R.$$

---

$$\int\sqrt{a^2+x^2}\,\mathrm{d}x = \left[\begin{array}{c} x=a\sinh t,\quad \mathrm{d}x=a\cosh t\,\mathrm{d}t,\quad x\in R,\ t\in R \\ \sqrt{a^2+x^2}=\sqrt{a^2+a^2\sinh^2 t}=a\sqrt{1+\sinh^2 t}=a\sqrt{\cosh^2 t}=a\,|\cosh t|=a\cosh t \end{array}\right]$$
$$= \int a^2\cosh^2 t\,\mathrm{d}t = a^2\int\left(\frac{\mathrm{e}^t+\mathrm{e}^{-t}}{2}\right)^2\mathrm{d}t = a^2\int\frac{\mathrm{e}^{2t}+2+\mathrm{e}^{-2t}}{4}\,\mathrm{d}t = a^2\int\left[\frac{\mathrm{e}^{2t}}{4}+\frac{1}{2}+\frac{\mathrm{e}^{-2t}}{4}\right]\mathrm{d}t$$
$$= a^2\left[\frac{\mathrm{e}^{2t}}{8}+\frac{t}{2}-\frac{\mathrm{e}^{-2t}}{8}\right]+c$$
$$= \left[\begin{array}{c} u=\mathrm{e}^t>0,\ x=a\sinh t=\frac{a}{2}(\mathrm{e}^t-\mathrm{e}^{-t})=\frac{a}{2}(u-u^{-1}),\ 2x=au-\frac{a}{u},\ au^2-2xu-a=0 \\ u_{1,2}=\frac{2x\pm\sqrt{4x^2+4a^2}}{2a}=\frac{2x\pm2\sqrt{a^2+x^2}}{2a}=\frac{x\pm\sqrt{a^2+x^2}}{a},\ \sqrt{a^2+x^2}>x\ \Rightarrow\ u=\mathrm{e}^t=\frac{x+\sqrt{a^2+x^2}}{a} \\ t=\ln\frac{x+\sqrt{a^2+x^2}}{a}=\ln\left(x+\sqrt{a^2+x^2}\right)-\ln a,\qquad \mathrm{e}^{2t}=(\mathrm{e}^t)^2=\frac{\left(x+\sqrt{a^2+x^2}\right)^2}{a^2} \\ \mathrm{e}^{-2t}=\frac{a^2}{\left(x+\sqrt{a^2+x^2}\right)^2}=\frac{a^2}{\left(x+\sqrt{a^2+x^2}\right)^2}\,\frac{\left(x-\sqrt{a^2+x^2}\right)^2}{\left(x-\sqrt{a^2+x^2}\right)^2}=\frac{a^2\left(x-\sqrt{a^2+x^2}\right)^2}{(x^2-a^2-x^2)^2}=\frac{\left(x-\sqrt{a^2+x^2}\right)^2}{a^2} \end{array}\right]$$
$$= a^2\left[\frac{\left(x+\sqrt{a^2+x^2}\right)^2}{8a^2}+\frac{\ln\left(x+\sqrt{a^2+x^2}\right)-\ln a}{2}-\frac{\left(x-\sqrt{a^2+x^2}\right)^2}{8a^2}\right]+c_1$$
$$= a^2\left[\frac{x^2+2x\sqrt{a^2+x^2}+a^2+x^2}{8a^2}+\frac{\ln\left(x+\sqrt{a^2+x^2}\right)-\ln a}{2}-\frac{x^2-2x\sqrt{a^2+x^2}+a^2+x^2}{8a^2}\right]+c_1$$
$$= a^2\left[\frac{4x\sqrt{a^2+x^2}}{8a^2}+\frac{\ln\left(x+\sqrt{a^2+x^2}\right)-\ln a}{2}\right]+c_1 = \frac{x\sqrt{a^2+x^2}}{2}+\frac{a^2\ln\left(x+\sqrt{a^2+x^2}\right)}{2}-\frac{a^2}{2}\ln a+c_1$$
$$= \left[\ c_1-\frac{a^2}{2}\ln a=c=\text{const.}\ \right] = \frac{x\sqrt{a^2+x^2}}{2}+\frac{a^2}{2}\ln\left(x+\sqrt{a^2+x^2}\right)+c,\ \text{for } a>0,\ x\in R.$$

---

$$\int\sqrt{a^2+x^2}\,\mathrm{d}x = \left[\begin{array}{c} \boxed{\text{1st ES}}\ \sqrt{a^2+x^2}=t-x,\ a^2+x^2=t^2-2tx+x^2,\ x=\frac{t^2-a^2}{2t},\ t=x+\sqrt{a^2+x^2} \\ \sqrt{a^2+x^2}=t-\frac{t^2-a^2}{2t}=\frac{t^2+a^2}{2t},\ \mathrm{d}x=\frac{2t\cdot 2t-2(t^2-a^2)}{4t^2}\,\mathrm{d}t=\frac{2t^2+2a^2}{4t^2}\,\mathrm{d}t=\frac{t^2+a^2}{2t^2}\,\mathrm{d}t \\ \frac{1}{t}=\frac{1}{x+\sqrt{a^2+x^2}}\frac{x-\sqrt{a^2+x^2}}{x-\sqrt{a^2+x^2}}=\frac{x-\sqrt{a^2+x^2}}{x^2-(a^2+x^2)}=\frac{x-\sqrt{a^2+x^2}}{a^2} \end{array}\right]$$

$$= \int\frac{t^2+a^2}{2t}\frac{t^2+a^2}{2t^2}\,\mathrm{d}t = \int\frac{t^4+2t^2a^2+a^4}{4t^3}\,\mathrm{d}t = \frac{1}{4}\int\left[t+2a^2t^{-1}+a^4t^{-3}\right]\mathrm{d}t$$

$$= \frac{1}{4}\left[\frac{t^2}{2}+2a^2\ln|t|+a^4\frac{t^{-2}}{-2}\right]+c = \frac{t^2}{8}-\frac{a^4}{2t^2}+2a^2\ln|t|+c$$

$$= \frac{\left(x+\sqrt{a^2+x^2}\right)^2}{8}-\frac{\left(x-\sqrt{a^2+x^2}\right)^2}{8}+\frac{a^2}{2}\ln\left|x+\sqrt{a^2+x^2}\right|+c$$

$$= \frac{x^2+2x\sqrt{a^2+x^2}+a^2+x^2}{8}-\frac{x^2-2x\sqrt{a^2+x^2}+a^2+x^2}{8}+\frac{a^2}{2}\ln\left(x+\sqrt{a^2+x^2}\right)+c$$

$$= \frac{4x\sqrt{a^2+x^2}}{8}+\frac{a^2}{2}\ln\left(x+\sqrt{a^2+x^2}\right)+c = \frac{x\sqrt{a^2+x^2}}{2}+\frac{a^2}{2}\ln\left(x+\sqrt{a^2+x^2}\right)+c,$$

for $a>0$, $x\in R$.

---

$$I = \int\sqrt{a^2+x^2}\,\mathrm{d}x = \int\frac{a^2+x^2}{\sqrt{a^2+x^2}}\,\mathrm{d}x = \int\frac{a^2\,\mathrm{d}x}{\sqrt{a^2+x^2}}+\int\frac{x^2\,\mathrm{d}x}{\sqrt{a^2+x^2}}$$

$$= \left[\begin{array}{l|l} u=x & u'=1 \\ v'=\frac{x}{\sqrt{a^2+x^2}}=\frac{2x}{2\sqrt{a^2+x^2}} & v=\sqrt{a^2+x^2} \end{array}\right] = \int\frac{a^2\,\mathrm{d}x}{\sqrt{a^2+x^2}}+\left[x\sqrt{a^2+x^2}-\int\sqrt{a^2+x^2}\,\mathrm{d}x\right]$$

$$= a^2\ln\left(x+\sqrt{a^2+x^2}\right)+x\sqrt{a^2+x^2}-\int\sqrt{a^2+x^2}\,\mathrm{d}x =$$

$$\left[\ \boxed{\text{Equation}}\ I=a^2\ln\left(x+\sqrt{a^2+x^2}\right)+x\sqrt{a^2+x^2}-I\ \Rightarrow\ 2I=a^2\ln\left(x+\sqrt{a^2+x^2}\right)+x\sqrt{a^2+x^2}\ \right]$$

$$= \frac{a^2}{2}\ln\left(x+\sqrt{a^2+x^2}\right)+\frac{x\sqrt{a^2+x^2}}{2}+c,$$ for $a>0$, $x\in R$.

---

$$\int\sqrt{a^2+x^2}\,\mathrm{d}x = \left[\begin{array}{c} x=a\,\mathrm{tg}\,t,\ t=\mathrm{arctg}\,\frac{x}{a},\ x\in R,\ t\in\left(-\frac{\pi}{2};\frac{\pi}{2}\right),\ \sqrt{\cos^2 x}=|\cos x|=\cos x \\ \mathrm{d}x=\frac{a\,\mathrm{d}t}{\cos^2 t},\ a^2+x^2=a^2+\frac{a^2\sin^2 t}{\cos^2 t}=\frac{a^2\cos^2 t+a^2\sin^2 t}{\cos^2 t}=\frac{a^2}{\cos^2 t},\ \sqrt{a^2+x^2}=\frac{a}{\cos t} \\ \mathrm{tg}\,t=\frac{\sin t}{\cos t},\ \sin t=\cos t\cdot\mathrm{tg}\,t=\frac{a}{\sqrt{a^2+x^2}}\frac{x}{a}=\frac{x}{\sqrt{a^2+x^2}},\ \sin t\pm 1=\frac{x\pm\sqrt{a^2+x^2}}{\sqrt{a^2+x^2}} \end{array}\right]$$

$$= \int\frac{a^2\,\mathrm{d}t}{\cos^3 t} = \int\frac{a^2\cos t\,\mathrm{d}t}{\cos^4 t} = \int\frac{a^2\cos t\,\mathrm{d}t}{(1-\sin^2 t)^2} = \left[\begin{array}{c} u=\sin t \\ \mathrm{d}u=\cos t\,\mathrm{d}t \end{array}\right] = \int\frac{a^2\,\mathrm{d}u}{(1-u^2)^2} = \int\frac{a^2\,\mathrm{d}u}{(u^2-1)^2}$$

$$= \int\frac{a^2\,\mathrm{d}u}{(u-1)^2(u+1)^2} = \left[\begin{array}{c} \frac{1}{(u-1)^2(u+1)^2}=\frac{A}{u+1}+\frac{B}{(u+1)^2}+\frac{C}{u-1}+\frac{D}{(u-1)^2} \\ A=\frac{1}{4},\ B=\frac{1}{4},\ C=-\frac{1}{4},\ D=\frac{1}{4} \end{array}\right]$$

$$= \frac{a^2}{4}\int\left[\frac{1}{u+1}+\frac{1}{(u+1)^2}-\frac{1}{u-1}+\frac{1}{(u-1)^2}\right]\mathrm{d}u = \frac{a^2}{4}\int\left[\frac{1}{u+1}-\frac{1}{u-1}+(u+1)^{-2}+(u-1)^{-2}\right]\mathrm{d}u$$

$$= \frac{a^2}{4}\left[\ln|u+1|-\ln|u-1|+\frac{(u+1)^{-1}}{-1}+\frac{(u-1)^{-1}}{-1}\right]+c_1 = \frac{a^2}{4}\left[\ln\left|\frac{u+1}{u-1}\right|-\frac{1}{u-1}-\frac{1}{u+1}\right]+c_1$$

$$= \frac{a^2}{4}\ln\left|\frac{u+1}{u-1}\right|-\frac{a^2(u+1+u-1)}{4(u^2-1)}+c_1 = \frac{a^2}{4}\ln\left|\frac{u+1}{u-1}\right|+\frac{ua^2}{2(1-u^2)}+c_1$$

$$= \frac{a^2}{4}\ln\left|\frac{\sin t+1}{\sin t-1}\right|+\frac{a^2\sin t}{2(1-\sin^2 t)}+c_1 = \frac{a^2}{4}\ln\left|\frac{\sin t+1}{\sin t-1}\right|+\frac{a^2}{\cos^2 t}\frac{\sin t}{2}+c_1$$

$$= \left[\ \frac{\sin t+1}{\sin t-1}=\frac{x+\sqrt{a^2+x^2}}{x-\sqrt{a^2+x^2}}\cdot\frac{x+\sqrt{a^2+x^2}}{x+\sqrt{a^2+x^2}}=\frac{\left(x+\sqrt{a^2+x^2}\right)^2}{x^2-(a^2+x^2)}=-\frac{\left(x+\sqrt{a^2+x^2}\right)^2}{a^2}\ \right]$$

$$= \frac{a^2}{4}\ln\left|\frac{\left(x+\sqrt{a^2+x^2}\right)^2}{a^2}\right|+(a^2+x^2)\frac{x}{2\sqrt{a^2+x^2}}+c_1 = \frac{a^2}{2}\ln\left|\frac{x+\sqrt{a^2+x^2}}{a}\right|+\frac{x\sqrt{a^2+x^2}}{2}+c_1$$

$$= \frac{a^2}{2}\ln\left|x+\sqrt{a^2+x^2}\right|-\frac{a^2}{2}\ln a+\frac{x\sqrt{a^2+x^2}}{2}+c_1 = \left[\ c_1-\frac{a^2}{2}\ln a=c=\text{const.}\ \right]$$

$$= \frac{a^2}{2}\ln\left(x+\sqrt{a^2+x^2}\right)+\frac{x\sqrt{a^2+x^2}}{2}+c,$$ for $a>0$, $x\in R$.

---

$$\int\sqrt{x^2-a^2}\,\mathrm{d}x = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + c, \text{ for } a>0,\ x\in(-\infty\,;\,-a\rangle\cup\langle a\,;\,\infty).$$

$$\int\sqrt{x^2-a^2}\,\mathrm{d}x = \left[\begin{array}{c} x=a\cosh t,\quad \mathrm{d}x=a\sinh t\,\mathrm{d}t,\quad x\in\langle a\,;\,\infty),\ t\in\langle 0\,;\,\infty) \\ \sqrt{x^2-a^2}=\sqrt{a^2\cosh^2 t-a^2}=a\sqrt{\cosh^2 t-1}=a\sqrt{\sinh^2 t}=a\,|\sinh t|=a\sinh t \end{array}\right]$$

$$= \int a^2\sinh^2 t\,\mathrm{d}t = a^2\int\Big(\frac{\mathrm{e}^t-\mathrm{e}^{-t}}{2}\Big)^2\mathrm{d}t = a^2\int\frac{\mathrm{e}^{2t}-2+\mathrm{e}^{-2t}}{4}\,\mathrm{d}t = a^2\int\Big[\frac{\mathrm{e}^{2t}}{4}-\frac{1}{2}+\frac{\mathrm{e}^{-2t}}{4}\Big]\mathrm{d}t$$

$$= a^2\Big[\frac{\mathrm{e}^{2t}}{8}-\frac{t}{2}-\frac{\mathrm{e}^{-2t}}{8}\Big]+c_1 = \frac{a^2}{4}\,\frac{\mathrm{e}^{2t}-\mathrm{e}^{-2t}}{2}-\frac{a^2t}{2}+c = \frac{a^2}{4}\sinh 2t-\frac{a^2t}{2}+c$$

$$= \left[\begin{array}{c} t=\operatorname{argcosh}\frac{x}{a}=\ln\Big(\frac{x}{a}+\sqrt{\frac{x^2}{a^2}-1}\Big)=\ln\frac{x+\sqrt{x^2-a^2}}{a}=\ln\big(x+\sqrt{x^2-a^2}\big)-\ln a \\ a^2\sinh 2t=2a\sinh t\cdot a\cosh t=2x\sqrt{x^2-a^2} \end{array}\right]$$

$$= \frac{x\sqrt{x^2-a^2}}{2}-\frac{a^2\big[\ln\big(x+\sqrt{x^2-a^2}\big)-\ln a\big]}{2}+c_1 = \frac{x\sqrt{x^2-a^2}}{2}-\frac{a^2}{2}\ln\big(x+\sqrt{x^2-a^2}\big)+c_1-\frac{a^2}{2}\ln a$$

$$= \Big[\ c_1-\frac{a^2}{2}\ln a=c=\text{const.}\ \Big] = \frac{x\sqrt{x^2-a^2}}{2}-\frac{a^2}{2}\ln\big(x+\sqrt{x^2-a^2}\big)+c, \text{ for } a>0,\ x\ge a.$$

$$\int\sqrt{x^2-a^2}\,\mathrm{d}x = \Big[\ x=-t,\ \mathrm{d}x=-\mathrm{d}t,\ x\le -a,\ t\ge a\ \Big] = -\int\sqrt{t^2-a^2}\,\mathrm{d}t$$

$$= -\frac{t\sqrt{t^2-a^2}}{2}+\frac{a^2}{2}\ln\big(t+\sqrt{t^2-a^2}\big)-c_2 = -\frac{-x\sqrt{x^2-a^2}}{2}+\frac{a^2}{2}\ln\big(-x+\sqrt{x^2-a^2}\big)-c_2$$

$$= \Big[\ -\big(-x+\sqrt{x^2-a^2}\big)=x-\sqrt{x^2-a^2}=\big(x-\sqrt{x^2-a^2}\big)\frac{x+\sqrt{x^2-a^2}}{x+\sqrt{x^2-a^2}}=\frac{x^2-(x^2-a^2)}{x+\sqrt{x^2-a^2}}=\frac{a^2}{x+\sqrt{x^2-a^2}}\ \Big]$$

$$= \frac{x\sqrt{x^2-a^2}}{2}+\frac{a^2}{2}\ln\Big|\frac{a^2}{x+\sqrt{x^2-a^2}}\Big|-c_2 = \frac{x\sqrt{x^2-a^2}}{2}+\frac{a^2}{2}2\ln a-\ln\big|x+\sqrt{x^2-a^2}\big|-c_2$$

$$= \Big[\ \frac{a^2}{2}\ln a-c_2=c=\text{const.}\ \Big] = \frac{x\sqrt{x^2-a^2}}{2}-\frac{a^2}{2}\ln\big|x+\sqrt{x^2-a^2}\big|+c, \text{ for } a>0,\ x\le -a.$$

---

$$\int\sqrt{x^2-a^2}\,\mathrm{d}x = \left[\begin{array}{c} \boxed{\text{1st ES}}\ \sqrt{x^2-a^2}=t-x,\ x^2-a^2=t^2-2tx+x^2,\ x=\frac{t^2+a^2}{2t},\ t=x+\sqrt{x^2-a^2} \\ \sqrt{x^2-a^2}=t-\frac{t^2+a^2}{2t}=\frac{t^2-a^2}{2t},\ \mathrm{d}x=\frac{2t\cdot 2t-2(t^2+a^2)}{4t^2}\,\mathrm{d}t=\frac{2t^2-2a^2}{4t^2}\,\mathrm{d}t=\frac{t^2-a^2}{2t^2}\,\mathrm{d}t \\ \frac{1}{t}=\frac{1}{x+\sqrt{x^2-a^2}}\,\frac{x-\sqrt{x^2-a^2}}{x-\sqrt{x^2-a^2}}=\frac{x-\sqrt{x^2-a^2}}{x^2-(x^2-a^2)}=\frac{x-\sqrt{x^2-a^2}}{a^2} \end{array}\right]$$

$$= \int\frac{t^2-a^2}{2t}\,\frac{t^2-a^2}{2t^2}\,\mathrm{d}t = \int\frac{t^4-2t^2a^2+a^4}{4t^3}\,\mathrm{d}t = \frac{1}{4}\int\big[t-2a^2t^{-1}+a^4t^{-3}\big]\,\mathrm{d}t$$

$$= \frac{1}{4}\Big[\frac{t^2}{2}-2a^2\ln|t|+a^4\frac{t^{-2}}{-2}\Big]+c = \frac{t^2}{8}-\frac{a^4}{2t^2}-2a^2\ln|t|+c$$

$$= \frac{\big(x+\sqrt{x^2-a^2}\big)^2}{8}-\frac{\big(x-\sqrt{x^2-a^2}\big)^2}{8}-\frac{a^2}{2}\ln\big|x+\sqrt{x^2-a^2}\big|+c$$

$$= \frac{x^2+2x\sqrt{x^2-a^2}+x^2-a^2}{8}-\frac{x^2-2x\sqrt{x^2-a^2}+x^2-a^2}{8}-\frac{a^2}{2}\ln\big|x+\sqrt{x^2-a^2}\big|+c$$

$$= \frac{4x\sqrt{x^2-a^2}}{8}-\frac{a^2}{2}\ln\big|x+\sqrt{x^2-a^2}\big|+c = \frac{x\sqrt{x^2-a^2}}{2}-\frac{a^2}{2}\ln\big|x+\sqrt{x^2-a^2}\big|+c,$$

for $a>0,\ x\in(-\infty\,;\,-a\rangle\cup\langle a\,;\,\infty)$.

---

$$I = \int\sqrt{x^2-a^2}\,\mathrm{d}x = \int\frac{x^2-a^2}{\sqrt{x^2-a^2}}\,\mathrm{d}x = \int\frac{x^2\,\mathrm{d}x}{\sqrt{x^2-a^2}}-\int\frac{a^2\,\mathrm{d}x}{\sqrt{x^2-a^2}} = \int\frac{x\cdot x\,\mathrm{d}x}{\sqrt{x^2-a^2}}-\int\frac{a^2\,\mathrm{d}x}{\sqrt{x^2-a^2}}$$

$$= \left[\begin{array}{l|l} u=x & u'=1 \\ v'=\frac{x}{\sqrt{x^2-a^2}}=\frac{2x}{2\sqrt{x^2-a^2}} & v=\sqrt{x^2-a^2} \end{array}\right] = \Big[x\sqrt{x^2-a^2}-\int\sqrt{x^2-a^2}\,\mathrm{d}x\Big]-\int\frac{a^2\,\mathrm{d}x}{\sqrt{x^2-a^2}}$$

$$= x\sqrt{x^2-a^2}-\int\sqrt{x^2-a^2}\,\mathrm{d}x-a^2\ln\big|x+\sqrt{x^2-a^2}\big| =$$

$$\Big[\ \boxed{\text{Equation}}\ I=x\sqrt{x^2-a^2}-I-a^2\ln\big|x+\sqrt{x^2-a^2}\big|\ \Rightarrow\ 2I=x\sqrt{x^2-a^2}-a^2\ln\big|x+\sqrt{x^2-a^2}\big|\ \Big]$$

$$= \frac{x\sqrt{x^2-a^2}}{2}-\frac{a^2}{2}\ln\big|x+\sqrt{x^2-a^2}\big|+c, \text{ for } a>0,\ x\in(-\infty\,;\,-a)\cup(a\,;\,\infty).$$

---

$$\int \operatorname{arctg} x \, dx = \left[ \begin{array}{l|l} u' = 1 & u = x \\ v = \operatorname{arctg} x & v' = \frac{1}{1+x^2} \end{array} \right] = x \operatorname{arctg} x - \int \frac{x}{1+x^2} \, dx = x \operatorname{arctg} x - \frac{1}{2} \int \frac{2x}{1+x^2} \, dx$$
$$= x \operatorname{arctg} x - \frac{1}{2} \ln \left|1 + x^2\right| + c = x \operatorname{arctg} x - \frac{1}{2} \ln \left(1 + x^2\right) + c$$
$$= x \operatorname{arctg} x - \ln \sqrt{1 + x^2} + c, \text{ for } x \in R.$$

---

$$I_n = \int x^n e^x \, dx = \left[ \begin{array}{l|l} u = x^n & u' = nx^{n-1} \\ v' = e^x & v = e^x \end{array} \right] = x^n e^x - n \int x^{n-1} e^x \, dx = x^n e^x - nI_{n-1},$$
for $n \in N$, $x \in R$.

$$I_0 = \int e^x \, dx = e^x + c, \text{ for } x \in R.$$

$$I_1 = \int x e^x \, dx = x e^x - I_0 = x e^x - 1 e^x + c, \text{ for } x \in R.$$

$$I_2 = \int x^2 e^x \, dx = x^2 e^x - 2I_1 = x^2 e^x - 2(x e^x - e^x) + c = x^2 e^x - 2x e^x + 2 \cdot 1 e^x + c,$$
for $x \in R$.

$$I_3 = \int x^3 e^x \, dx = x^3 e^x - 3I_2 = x^3 e^x - 3(x^2 e^x - 2x e^x + 2 \cdot 1 e^x) + c$$
$$= x^3 e^x - 3x^2 e^x + 3 \cdot 2x e^x - 3 \cdot 2 \cdot 1 e^x + c, \text{ for } x \in R.$$

$$I_4 = \int x^4 e^x \, dx = x^4 e^x - 4I_3 = x^4 e^x - 4(x^3 e^x - 3x^2 e^x + 3 \cdot 2x e^x - 3 \cdot 2 \cdot 1 e^x) + c$$
$$= x^4 e^x - 4x^3 e^x + 4 \cdot 3x^2 e^x - 4 \cdot 3 \cdot 2x e^x + 4 \cdot 3 \cdot 2 \cdot 1 e^x + c, \text{ for } x \in R.$$

...

$$I_n = \int x^n e^x \, dx = x^n e^x - nI_{n-1} = \sum_{j=0}^{n} \frac{(-1)^j x^{n-j} e^x n!}{(n-j)!} + c = e^x \sum_{j=0}^{n} \frac{(-1)^j x^{n-j} n!}{(n-j)!} + c$$
$$= e^x \sum_{j=0}^{n} (-1)^j n(n-1) \cdots (n-j+1) x^{n-j} + c$$
$$= e^x \left[ x^n - nx^{n-1} + n(n-1)x^{n-2} - n(n-1)(n-2)x^{n-3} + \cdots + (-1)^n n! \right] + c,$$
for $n \in N$, $x \in R$.

---

$$\int \frac{dx}{\sqrt{x+1} + \sqrt[3]{x+1}} = \int \frac{dx}{\left(\sqrt[6]{x+1}\right)^3 + \left(\sqrt[6]{x+1}\right)^2} = \left[ \begin{array}{c} \sqrt[6]{x+1} = t, \; x+1 = t^6 \\ dx = 6t^5 \, dt, \; x > -1, \; t > 0 \end{array} \right] = \int \frac{6t^5 \, dt}{t^3 + t^2} = 6 \int \frac{t^3 \, dt}{t+1}$$
$$= 6 \int \frac{t^3 + t^2 - t^2 - t + t + 1 - 1}{t+1} \, dt = 6 \int \left[ \frac{t^3 + t^2}{t+1} - \frac{t^2 + t}{t+1} + \frac{t+1}{t+1} - \frac{1}{t+1} \right] dt$$
$$= 6 \int \left[ t^2 - t + 1 - \frac{1}{t+1} \right] dt = 6 \left[ \frac{t^3}{3} - \frac{t^2}{2} + t - \ln |t + 1| \right] + c$$
$$= 2 \left( \sqrt[6]{x+1} \right)^3 - 3 \left( \sqrt[6]{x+1} \right)^2 + 6 \sqrt[6]{x+1} - 6 \ln \left| 1 + \sqrt[6]{x+1} \right| + c$$
$$= 2\sqrt{x+1} - 3\sqrt[3]{x+1} + 6\sqrt[6]{x+1} - 6 \ln \left| 1 + \sqrt[6]{x+1} \right| + c, \text{ for } x > -1.$$

---

$$\int \ln \left(1 + x^2\right) dx = \left[ \begin{array}{l|l} u' = 1 & u = x \\ v = \ln \left(1 + x^2\right) & v' = \frac{2x}{1+x^2} \end{array} \right] = x \ln \left(1 + x^2\right) - \int \frac{2x^2 \, dx}{1+x^2}$$
$$= x \ln \left(1 + x^2\right) - 2 \int \frac{x^2 + 1 - 1}{1+x^2} \, dx = x \ln \left(1 + x^2\right) - 2 \int \left[ 1 - \frac{1}{1+x^2} \right] dx$$
$$= x \ln \left(1 + x^2\right) - 2x + 2 \operatorname{arctg} x + c, \text{ for } x \in R.$$

---

$$\int\frac{dx}{\sqrt{a^2+x^2}}=\left[\begin{array}{c}\boxed{\text{1st ES}}\ \sqrt{a^2+x^2}=t-x,\ a^2+x^2=t^2-2tx+x^2,\ x=\frac{t^2-a^2}{2t},\ t=x+\sqrt{a^2+x^2}>0\\ \sqrt{a^2+x^2}=t-\frac{t^2-a^2}{2t}=\frac{t^2+a^2}{2t},\ dx=\frac{2t\cdot 2t-2(t^2-a^2)}{4t^2}\,dt=\frac{2t^2+2a^2}{4t^2}\,dt=\frac{t^2+a^2}{2t^2}\,dt\end{array}\right]$$
$$=\int\frac{2t}{t^2+a^2}\frac{t^2+a^2}{2t^2}\,dt=\int\frac{dt}{t}=\ln|t|+c=\ln\left|x+\sqrt{a^2+x^2}\right|+c$$
$$=\ln\left(x+\sqrt{a^2+x^2}\right)+c,\ \text{for } a>0,\ x\in R.$$

---

$$\int\frac{dx}{\sqrt{a^2+x^2}}=\left[\begin{array}{c}x=a\,\mathrm{tg}\,t,\ t=\mathrm{arctg}\,\frac{x}{a},\ x\in R,\ t\in\left(-\frac{\pi}{2};\frac{\pi}{2}\right),\ \sqrt{\cos^2 x}=|\cos x|=\cos x\\ dx=\frac{a\,dt}{\cos^2 t},\ a^2+x^2=a^2+\frac{a^2\sin^2 t}{\cos^2 t}=\frac{a^2\cos^2 t+a^2\sin^2 t}{\cos^2 t}=\frac{a^2}{\cos^2 t},\ \sqrt{a^2+x^2}=\frac{a}{\cos t}\\ \mathrm{tg}\,t=\frac{\sin t}{\cos t},\ \sin t=\cos t\cdot\mathrm{tg}\,t=\frac{a}{\sqrt{a^2+x^2}}\frac{x}{a}=\frac{x}{\sqrt{a^2+x^2}},\ \sin t\pm 1=\frac{x\pm\sqrt{a^2+x^2}}{\sqrt{a^2+x^2}}\end{array}\right]$$
$$=\int\frac{\cos t}{a}\frac{a\,dt}{\cos^2 t}=\int\frac{\cos t\,dt}{1-\sin^2 t}=\left[\begin{array}{c}u=\sin t\\ du=\cos t\,dt\end{array}\right]=\int\frac{du}{1-u^2}=-\int\frac{du}{u^2-1}=-\frac{1}{2}\ln\left|\frac{u-1}{u+1}\right|+c_1$$
$$=-\frac{1}{2}\ln\left|\frac{\sin t-1}{\sin t+1}\right|+c_1=\left[\frac{\sin t-1}{\sin t+1}=\frac{x-\sqrt{a^2+x^2}}{x+\sqrt{a^2+x^2}}\cdot\frac{x+\sqrt{a^2+x^2}}{x+\sqrt{a^2+x^2}}=\frac{x^2-(a^2+x^2)}{\left(x+\sqrt{a^2+x^2}\right)^2}=\frac{-a^2}{\left(x+\sqrt{a^2+x^2}\right)^2}\right]$$
$$=-\frac{1}{2}\ln\left|\frac{a^2}{\left(x+\sqrt{a^2+x^2}\right)^2}\right|+c_1=-\frac{2}{2}\ln\left|\frac{a}{x+\sqrt{a^2+x^2}}\right|+c_1=-\ln a+\ln\left|x+\sqrt{a^2+x^2}\right|+c_1$$
$$=\left[\ x+\sqrt{a^2+x^2}>0\ \middle|\ c_1-\ln a=c=\text{const.}\ \right]=\ln\left(x+\sqrt{a^2+x^2}\right)+c,\ \text{for } a>0,\ x\in R.$$

---

$$\int\frac{dx}{\sqrt{x^2-a^2}}=\left[\begin{array}{c}\boxed{\text{1st ES}}\ \sqrt{x^2-a^2}=t-x,\ x^2-a^2=t^2-2tx+x^2,\ x=\frac{t^2+a^2}{2t},\ t=x+\sqrt{x^2-a^2}\\ \sqrt{x^2-a^2}=t-\frac{t^2+a^2}{2t}=\frac{t^2-a^2}{2t},\ dx=\frac{2t\cdot 2t-2(t^2+a^2)}{4t^2}\,dt=\frac{2t^2-2a^2}{4t^2}\,dt=\frac{t^2-a^2}{2t^2}\,dt\end{array}\right]$$
$$=\int\frac{2t}{t^2-a^2}\frac{t^2-a^2}{2t^2}\,dt=\int\frac{dt}{t}=\ln|t|+c=\ln\left|x+\sqrt{x^2-a^2}\right|+c,$$
for $a>0,\ x\in(-\infty;-a)\cup(a;\infty)$.

---

$$\int\frac{dx}{\sqrt{a^2-x^2}}=\left[\begin{array}{c}x=a\sin t,\ t=\arcsin\frac{x}{a},\ dx=a\cos t\,dt,\ x\in(-a;a),\ t\in\left(-\frac{\pi}{2};\frac{\pi}{2}\right)\\ \sqrt{a^2-x^2}=\sqrt{a^2-a^2\sin^2 t}=a\sqrt{1-\sin^2 t}=a\sqrt{\cos^2 t}=a|\cos t|=a\cos t\end{array}\right]$$
$$=\int\frac{a\cos t\,dt}{a\cos t}=\int dt=t+c=\arcsin\frac{x}{a}+c,\ \text{for } a>0,\ x\in(-a;a).$$

---

$$\int\frac{dx}{\sqrt{a^2-x^2}}=\left[\begin{array}{c}x=a\cos t,\ t=\arccos\frac{x}{a},\ dx=-a\sin t\,dt,\ x\in(-a;a),\ t\in(0;\pi)\\ \sqrt{a^2-x^2}=\sqrt{a^2-a^2\cos^2 t}=a\sqrt{1-\cos^2 t}=a\sqrt{\sin^2 t}=a|\sin t|=a\sin t\end{array}\right]$$
$$=-\int\frac{a\sin t\,dt}{a\sin t}=-\int dt=-t+c=-\arccos\frac{x}{a}+c,\ \text{for } a>0,\ x\in(-a;a).$$

---

$$\int\frac{dx}{\sqrt{a^2-x^2}}=\left[\begin{array}{c}\boxed{\text{2nd ES}}\ \sqrt{a^2-x^2}=a-xt,\ t=\frac{a-\sqrt{a^2-x^2}}{x},\ x\in(-a;a),\ x\neq 0,\ t\in(-1;1),\ t\neq 0\\ a^2-x^2=x^2t^2-2axt+a^2,\ 2axt=x^2+x^2t^2,\ 2at=x+xt^2,\ x=\frac{2at}{1+t^2}\\ dx=\frac{2a(1+t^2)-2at\cdot 2t}{(1+t^2)^2}\,dt=\frac{2a(1-t^2)}{(1+t^2)^2}\,dt,\ \sqrt{a^2-x^2}=a-xt=a-\frac{2at^2}{1+t^2}=\frac{a-at^2}{1+t^2}=\frac{a(1-t^2)}{1+t^2}\end{array}\right]$$
$$=\int\frac{1+t^2}{a(1-t^2)}\frac{2a(1-t^2)}{(1+t^2)^2}\,dt=2\int\frac{dt}{t^2+1}=2\,\mathrm{arctg}\,t+c=2\,\mathrm{arctg}\,\frac{a-\sqrt{a^2-x^2}}{x}+c,$$
for $a>0,\ x\in(-a;a)-\{0\}$.

---

$$\int x^2\,\mathrm{e}^{2x}\,\mathrm{d}x = \left[\begin{array}{l|l} u = x^2 & u' = 2x \\ v' = \mathrm{e}^{2x} & v = \frac{\mathrm{e}^{2x}}{2} \end{array}\right] = \frac{x^2\,\mathrm{e}^{2x}}{2} - \int \frac{2x\,\mathrm{e}^{2x}}{2}\,\mathrm{d}x = \frac{x^2\,\mathrm{e}^{2x}}{2} - \int x\,\mathrm{e}^{2x}\,\mathrm{d}x$$

$$= \left[\begin{array}{l|l} u = x & u' = 1 \\ v' = \mathrm{e}^{2x} & v = \frac{\mathrm{e}^{2x}}{2} \end{array}\right] = \frac{x^2\,\mathrm{e}^{2x}}{2} - \left[\frac{x\,\mathrm{e}^{2x}}{2} - \int \frac{\mathrm{e}^{2x}}{2}\,\mathrm{d}x\right] = \frac{x^2\,\mathrm{e}^{2x}}{2} - \frac{x\,\mathrm{e}^{2x}}{2} + \int \frac{\mathrm{e}^{2x}}{2}\,\mathrm{d}x$$

$$= \frac{x^2\,\mathrm{e}^{2x}}{2} - \frac{x\,\mathrm{e}^{2x}}{2} + \frac{\mathrm{e}^{2x}}{4} + c = \mathrm{e}^{2x}\left[\frac{x^2}{2} - \frac{x}{2} + \frac{1}{4}\right] + c, \text{ for } x \in R.$$

---

$$\int x^2\,\mathrm{e}^{2x}\,\mathrm{d}x = \mathrm{e}^{2x}\left[Ax^2 + Bx + C\right] + c = \mathrm{e}^{2x}\left[\frac{x^2}{2} - \frac{x}{2} + \frac{1}{4}\right] + c, \text{ for } x \in R.$$

Derivative

$$x^2\,\mathrm{e}^{2x} = \left[\int x^2\,\mathrm{e}^{2x}\,\mathrm{d}x\right]' = \left[\mathrm{e}^{2x}(Ax^2+Bx+C)+c\right]' = 2\,\mathrm{e}^{2x}(Ax^2+Bx+C)+\mathrm{e}^{2x}(2Ax+B)$$

Equations

$$0 + 0x + x^2 = (2C+B) + (2B+2A)x + 2Ax^2$$

$$1 = 2A,\ 0 = 2A+2B,\ 0 = B+2C, \quad A = \tfrac{1}{2},\ B = -\tfrac{1}{2},\ C = \tfrac{1}{4}$$

---

$$\int x^9\,\mathrm{e}^{2x}\,\mathrm{d}x = \left[\begin{array}{l|l} u = x^9 & u' = 9x^8 \\ v' = \mathrm{e}^{2x} & v = \frac{\mathrm{e}^{2x}}{2} \end{array}\right] = \frac{x^9\,\mathrm{e}^{2x}}{2} - \int \frac{9x^8\,\mathrm{e}^{2x}}{2}\,\mathrm{d}x = \left[\begin{array}{l|l} u = x^8 & u' = 8x^7 \\ v' = \mathrm{e}^{2x} & v = \frac{\mathrm{e}^{2x}}{2} \end{array}\right] = \cdots ☹$$

---

$$\int x^9\,\mathrm{e}^{2x}\,\mathrm{d}x = \mathrm{e}^{2x}\left[Ax^9 + Bx^8 + Cx^7 + Dx^6 + Ex^5 + Fx^4 + Gx^3 + Hx^2 + Ix + J\right] + c$$

Derivative

$$x^9\,\mathrm{e}^{2x} = 2\,\mathrm{e}^{2x}(Ax^9+Bx^8+Cx^7+Dx^6+Ex^5+Fx^4+Gx^3+Hx^2+Ix+J) + \mathrm{e}^{2x}(9Ax^8+8Bx^7+7Cx^6+6Dx^5+5Ex^4+4Fx^3+3Gx^2+2Hx+I)$$

Equations

$$x^9 = 2Ax^9 + (2B+9A)x^8 + (2C+8B)x^7 + (2D+7C)x^6 + (2E+6D)x^5 + (2F+5E)x^4 + (2G+4F)x^3 + (2H+3G)x^2 + (2I+2H)x + (2J+I)$$

$$A=\tfrac{1}{2},\ B=-\tfrac{9}{4},\ C=9,\ D=-\tfrac{63}{2},\ E=\tfrac{189}{2},\ F=-\tfrac{945}{4},\ G=\tfrac{945}{2},\ H=-\tfrac{2835}{4},\ I=\tfrac{2835}{4},\ J=-\tfrac{2835}{8}$$

$$= \mathrm{e}^{2x}\left[\frac{x^9}{2} - \frac{9x^8}{4} + 9x^7 - \frac{63x^6}{2} + \frac{189x^5}{2} - \frac{945x^4}{4} + \frac{945x^3}{2} - \frac{2835x^2}{4} + \frac{2835x}{4} - \frac{2835}{8}\right] + c, \text{ for } x \in R.$$

---

$$\int \frac{\sqrt{1-\sqrt{x}}}{\sqrt{1+\sqrt{x}}}\,\mathrm{d}x = \int \frac{\sqrt{1-\sqrt{x}}}{\sqrt{1+\sqrt{x}}} \cdot \frac{\sqrt{1-\sqrt{x}}}{\sqrt{1-\sqrt{x}}}\,\mathrm{d}x = \left[\begin{array}{c} 1-\sqrt{x} > 0 \\ x \in \langle 0\,;\,1) \end{array}\right] = \int \frac{1-\sqrt{x}}{\sqrt{1-x}}\,\mathrm{d}x = \int \frac{\mathrm{d}x}{\sqrt{1-x}} - \int \frac{\sqrt{x}}{\sqrt{1-x}}\,\mathrm{d}x$$

$$= \int (1-x)^{-\frac{1}{2}}\,\mathrm{d}x - \int \sqrt{\frac{x}{1-x}}\,\mathrm{d}x = \left[\begin{array}{c|l} 1-x = u & \frac{x}{1-x} = t^2,\ x = t^2 - xt^2,\ x = \frac{t^2}{1+t^2} \\ \mathrm{d}x = -\,\mathrm{d}u & \mathrm{d}x = \frac{2t(1+t^2)-t^2\cdot 2t}{(1+t^2)^2}\,\mathrm{d}t = \frac{2t\,\mathrm{d}t}{(1+t^2)^2} \end{array}\right]$$

$$= -\int u^{-\frac{1}{2}}\,\mathrm{d}u - \int \frac{t\cdot 2t\,\mathrm{d}t}{(1+t^2)^2} = -\frac{u^{\frac{1}{2}}}{\frac{1}{2}} - 2\int \frac{t^2\,\mathrm{d}t}{(1+t^2)^2} = -2\sqrt{u} - 2\int \frac{1+t^2-1}{(1+t^2)^2}\,\mathrm{d}t$$

$$= -2\sqrt{1-x} - 2\int \left[\frac{1}{1+t^2} - \frac{1}{(1+t^2)^2}\right]\mathrm{d}t = \left[\text{p. 21}: \int \frac{\mathrm{d}t}{(1+t^2)^2} = \frac{1}{2}\,\mathrm{arctg}\,t + \frac{t}{2(t^2+1)},\ t \in R\right]$$

$$= -2\sqrt{1-x} - 2\left[\mathrm{arctg}\,t - \frac{1}{2}\,\mathrm{arctg}\,t - \frac{1}{2}\,\frac{t}{t^2+1}\right] + c = -2\sqrt{1-x} - \mathrm{arctg}\,t + \frac{t}{t^2+1} + c$$

$$= \left[\begin{array}{l} t^2 + 1 = \frac{x}{1-x} + 1 = \frac{x+1-x}{1-x} = \frac{1}{1-x} \\ \frac{t}{t^2+1} = (1-x)\sqrt{\frac{x}{1-x}} = \sqrt{(1-x)x} \end{array}\right] = -2\sqrt{1-x} - \mathrm{arctg}\,\sqrt{\frac{x}{1-x}} + \sqrt{x(1-x)} + c,$$

for $x \in \langle 0\,;\,1)$.

---

$$\int |x|\,\mathrm{d}x = \begin{cases} \int x\,\mathrm{d}x = \frac{x^2}{2} + c = \frac{x\,|x|}{2} + c, & \text{for } x \geq 0, \\ -\int x\,\mathrm{d}x = -\frac{x^2}{2} + c = \frac{x\,|x|}{2} + c, & \text{for } x \leq 0. \end{cases}$$

---

$$\int \ln x\,\mathrm{d}x = \left[\begin{array}{l|l} u = \ln x & u' = \frac{1}{x} \\ v' = 1 & v = x \end{array}\right] = x\ln x - \int x\frac{\mathrm{d}x}{x} = x\ln x - \int \mathrm{d}x = x\ln x - x + c, \text{ for } x > 0.$$

---

$$\int x^x(\ln x + 1)\,\mathrm{d}x = \left[\begin{array}{l|l} u = x^x & u' = \left[\mathrm{e}^{\ln x^x}\right]' = \left[\mathrm{e}^{x\ln x}\right]' = \mathrm{e}^{x\ln x}\left[\ln x + \frac{x}{x}\right] = x^x(\ln x + 1) \\ v' = 1 & v = x \end{array}\right]$$
$$= x^x + c, \text{ for } x > 0.$$

---

$$\int \ln\left(x+\sqrt{x^2+1}\right)\mathrm{d}x = \left[\begin{array}{l|l} u = \ln\left(x+\sqrt{x^2+1}\right) & u' = \frac{1}{\sqrt{x^2+1}} \\ v' = 1 & v = x \end{array}\right] = x\ln\left(x+\sqrt{x^2+1}\right) - \int \frac{x\,\mathrm{d}x}{\sqrt{x^2+1}}$$
$$= \left[\begin{array}{c} x^2+1 = t \\ 2x\,\mathrm{d}x = \mathrm{d}t \end{array}\right] = x\ln\left(x+\sqrt{x^2+1}\right) - \frac{1}{2}\int \frac{\mathrm{d}t}{\sqrt{t}} = x\ln\left(x+\sqrt{x^2+1}\right) - \frac{1}{2}\int t^{-\frac{1}{2}}\,\mathrm{d}t$$
$$= x\ln\left(x+\sqrt{x^2+1}\right) - \frac{1}{2}\frac{t^{\frac{1}{2}}}{\frac{1}{2}} + c = x\ln\left(x+\sqrt{x^2+1}\right) - \sqrt{x^2+1} + c, \text{ for } x \in R.$$

---

$$\int \frac{\mathrm{d}x}{\sqrt{x-3}+\sqrt{x-4}} = \int \frac{1}{\sqrt{x-3}+\sqrt{x-4}}\,\frac{\sqrt{x-3}-\sqrt{x-4}}{\sqrt{x-3}-\sqrt{x-4}}\,\mathrm{d}x = \int \frac{\sqrt{x-3}-\sqrt{x-4}}{x-3-(x-4)}\,\mathrm{d}x$$
$$= \int\left[\sqrt{x-3} - \sqrt{x-4}\right]\mathrm{d}x = \int\left[(x-3)^{\frac{1}{2}} - (x-4)^{\frac{1}{2}}\right]\mathrm{d}x = \frac{(x-3)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{(x-4)^{\frac{3}{2}}}{\frac{3}{2}} + c$$
$$= \frac{2\sqrt{(x-3)^3}}{3} - \frac{2\sqrt{(x-4)^3}}{3} + c, \text{ for } x > 3.$$

---

$$\int \min_{x\in(0;\,\infty)}\left\{1, \tfrac{1}{x}\right\}\mathrm{d}x = \begin{cases} \int \mathrm{d}x = x + c_1, & \text{for } x \in (0\,;\,1\rangle, \\ \int \frac{\mathrm{d}x}{x} = \ln x + c_2, & \text{for } x \in \langle 1\,;\,\infty). \end{cases}$$

---

$$\int \frac{\mathrm{d}x}{x^2\sqrt{x^2-1}} = \left[\begin{array}{l} \boxed{\text{1st ES}}\ \sqrt{x^2-1} = t-x,\ x^2-1 = t^2-2tx+x^2,\ x = \frac{t^2+1}{2t},\ t = x+\sqrt{x^2-1} \\ \sqrt{x^2-1} = t-\frac{t^2+1}{2t} = \frac{t^2-1}{2t},\ \mathrm{d}x = \frac{2t\cdot 2t-2(t^2+1)}{4t^2}\,\mathrm{d}t = \frac{2t^2-2}{4t^2}\,\mathrm{d}t = \frac{t^2-1}{2t^2}\,\mathrm{d}t \end{array}\right]$$
$$= \int\left(\frac{2t}{t^2+1}\right)^2 \frac{2t}{t^2-1}\,\frac{t^2-1}{2t^2}\,\mathrm{d}t = \int \frac{4t\,\mathrm{d}t}{(t^2+1)^2} = \left[\begin{array}{c} t^2+1 = u \\ 2t\,\mathrm{d}t = \mathrm{d}u \end{array}\right] = \int \frac{2\,\mathrm{d}u}{u^2} = 2\int u^{-2}\,\mathrm{d}u$$
$$= 2\frac{u^{-1}}{-1} + c_1 = c_1 - \frac{2}{u} = c_1 - \frac{2}{t^2+1} = c_1 - \frac{2}{2tx} = c_1 - \frac{1}{x\left(x+\sqrt{x^2-1}\right)}$$
$$= c_1 - \frac{1}{x\left(x+\sqrt{x^2-1}\right)}\cdot\frac{x-\sqrt{x^2-1}}{x-\sqrt{x^2-1}} = c_1 - \frac{x-\sqrt{x^2-1}}{x(x^2-x^2+1)} = c_1 - \frac{x-\sqrt{x^2-1}}{x} = c_1 - 1 + \frac{\sqrt{x^2-1}}{x}$$
$$= \left[\ c_1 - 1 = c = \text{const.}\ \right] = \frac{\sqrt{x^2-1}}{x} + c, \text{ for } x \in (-\infty\,;\,-1) \cup (1\,;\,\infty).$$

---

$\int \frac{\mathrm{d}x}{x^2\sqrt{x^2-1}} = \frac{\sqrt{x^2-1}}{x} + c$, for $x \in (-\infty\,;\,-1) \cup (1\,;\,\infty)$.

$$\int \frac{\mathrm{d}x}{x^2\sqrt{x^2-1}} = \left[\begin{array}{c} x=\frac{1}{t},\ t=\frac{1}{x},\ \mathrm{d}x=-\frac{\mathrm{d}t}{t^2},\ x\in(1\,;\,\infty),\ t\in(0\,;\,1) \\ \sqrt{x^2-1}=\sqrt{\frac{1}{t^2}-1}=\frac{\sqrt{1-t^2}}{t},\ \sqrt{1-t^2}=\sqrt{1-\frac{1}{x^2}}=\frac{\sqrt{x^2-1}}{x} \end{array}\right] = -\int \frac{t^2\cdot t}{\sqrt{1-t^2}}\frac{\mathrm{d}t}{t^2}$$

$$= \int \frac{-t\,\mathrm{d}t}{\sqrt{1-t^2}} = \left[\begin{array}{c} 1-t^2=u \\ -2t\,\mathrm{d}t=\mathrm{d}u \end{array}\right] = \frac{1}{2}\int \frac{\mathrm{d}u}{\sqrt{u}} = \frac{1}{2}\int u^{-\frac{1}{2}}\,\mathrm{d}u = \frac{1}{2}\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c = \sqrt{1-t^2} + c$$

$= \sqrt{1-\frac{1}{x^2}} + c = \frac{\sqrt{x^2-1}}{x} + c$, for $x \in (1\,;\,\infty)$.

$$\int \frac{\mathrm{d}x}{x^2\sqrt{x^2-1}} = \left[\begin{array}{c} x=-t,\ \mathrm{d}x=-\mathrm{d}t \\ x\in(-\infty\,;\,-1),\ t\in(1\,;\,\infty) \end{array}\right] = -\int \frac{\mathrm{d}t}{t^2\sqrt{t^2-1}} = -\frac{\sqrt{t^2-1}}{t} + c = -\frac{\sqrt{x^2-1}}{-x} + c$$

$= \frac{\sqrt{x^2-1}}{x} + c$, for $x \in (-\infty\,;\,-1)$.

---

$$\int \frac{2x^2-x+1}{x\sqrt{1+x-x^2}}\,\mathrm{d}x = \left[\begin{array}{c} x^2-x-1=(x-\frac{1}{2})^2-\frac{5}{4}=\left(x-\frac{1}{2}-\frac{\sqrt{5}}{2}\right)\left(x-\frac{1}{2}+\frac{\sqrt{5}}{2}\right)<0 \\ x\in\left(\frac{1}{2}-\frac{\sqrt{5}}{2};\,\frac{1}{2}+\frac{\sqrt{5}}{2}\right),\ x\neq 0\ \Rightarrow\ x\in\left(\frac{1}{2}-\frac{\sqrt{5}}{2};\,0\right)\cup\left(0;\,\frac{1}{2}+\frac{\sqrt{5}}{2}\right) \end{array}\right]$$

$$= \int \frac{(2x-1)\,\mathrm{d}x}{\sqrt{1+x-x^2}} + \int \frac{\mathrm{d}x}{x\sqrt{1+x-x^2}} = -2\sqrt{1+x-x^2} - \ln\left|\frac{2+x+2\sqrt{1+x-x^2}}{x}\right| + c,$$

for $x \in \left(\frac{1}{2}-\frac{\sqrt{5}}{2};\,0\right)\cup\left(0;\,\frac{1}{2}+\frac{\sqrt{5}}{2}\right)$.

$$\int \frac{(2x-1)\,\mathrm{d}x}{\sqrt{1+x-x^2}} = \left[\begin{array}{c} 1+x-x^2=s \\ (1-2x)\,\mathrm{d}x=\mathrm{d}s \end{array}\right] = \int \frac{-\mathrm{d}s}{\sqrt{s}} = -\int s^{-\frac{1}{2}}\,\mathrm{d}s = -\frac{s^{\frac{1}{2}}}{\frac{1}{2}} + c_1 = -2\sqrt{s} + c_1$$

$= -2\sqrt{1+x-x^2} + c_1$, for $x \in \left(\frac{1}{2}-\frac{\sqrt{5}}{2};\,\frac{1}{2}+\frac{\sqrt{5}}{2}\right)$.

$$\int \frac{\mathrm{d}x}{x\sqrt{1+x-x^2}} = \left[\begin{array}{c} x=\frac{1}{t},\ t=\frac{1}{x},\ \mathrm{d}x=-\frac{\mathrm{d}t}{t^2},\ \sqrt{1+x-x^2}=\sqrt{1+\frac{1}{t}-\frac{1}{t^2}}=\frac{\sqrt{t^2+t-1}}{|t|}=\frac{\sqrt{t^2+t-1}}{t} \\ \sqrt{t^2+t-1}=\sqrt{\frac{1}{x^2}+\frac{1}{x}-1}=\frac{\sqrt{1+x-x^2}}{|x|}=\frac{\sqrt{1+x-x^2}}{x},\ 0<x<\frac{1+\sqrt{5}}{2} \\ t=\frac{1}{x}>\frac{2}{1+\sqrt{5}}=\frac{2}{1+\sqrt{5}}\frac{1-\sqrt{5}}{1-\sqrt{5}}=\frac{2(1-\sqrt{5})}{1-5}=\frac{2(1-\sqrt{5})}{-4}=\frac{\sqrt{5}-1}{2} \end{array}\right]$$

$$= \int \frac{t\cdot t}{\sqrt{t^2+t-1}}\frac{-\mathrm{d}t}{t^2} = -\int \frac{\mathrm{d}t}{\sqrt{t^2+t-1}} = \left[\begin{array}{c} t+\frac{1}{2}=u,\ \mathrm{d}t=\mathrm{d}u,\ u>\frac{\sqrt{5}}{2} \\ t^2+t-1=(t+\frac{1}{2})^2-\frac{5}{4}=u^2-\frac{5}{4} \end{array}\right] = -\int \frac{\mathrm{d}u}{\sqrt{u^2-\frac{5}{4}}}$$

$$= -\ln\left|u+\sqrt{u^2-\tfrac{5}{4}}\right|+c_2 = -\ln\left|t+\tfrac{1}{2}+\sqrt{t^2+t-1}\right|+c_2 = -\ln\left|\frac{1}{x}+\frac{1}{2}+\frac{\sqrt{1+x-x^2}}{x}\right|+c_2$$

$$= -\ln\left|\frac{2+x+2\sqrt{1+x-x^2}}{2x}\right| + c_2 = -\ln\left|\frac{2+x+2\sqrt{1+x-x^2}}{x}\right| + \ln 2 + c_2 = \left[\ \ln 2 + c_2 = c_3\ \right]$$

$= -\ln\left|\frac{2+x+2\sqrt{1+x-x^2}}{x}\right| + c_3$, for $x \in \left(0;\,\frac{1+\sqrt{5}}{2}\right)$.

$$\int \frac{\mathrm{d}x}{x\sqrt{1+x-x^2}} = \left[\begin{array}{c} x=\frac{1}{t},\ t=\frac{1}{x},\ \mathrm{d}x=-\frac{\mathrm{d}t}{t^2},\ \sqrt{1+x-x^2}=\frac{\sqrt{t^2+t-1}}{|t|}=\frac{\sqrt{t^2+t-1}}{-t} \\ \sqrt{t^2+t-1}=\frac{\sqrt{1+x-x^2}}{|x|}=\frac{\sqrt{1+x-x^2}}{-x},\ 0>x>\frac{1-\sqrt{5}}{2} \\ t=\frac{1}{x}<\frac{2}{1-\sqrt{5}}\frac{1+\sqrt{5}}{1+\sqrt{5}}=\frac{2(1+\sqrt{5})}{1-5}=\frac{2(1+\sqrt{5})}{-4}=-\frac{1+\sqrt{5}}{2} \end{array}\right] = \int \frac{-t\cdot t}{\sqrt{t^2+t-1}}\frac{-\mathrm{d}t}{t^2}$$

$$= \int \frac{\mathrm{d}t}{\sqrt{t^2+t-1}} = \left[\begin{array}{c} t+\frac{1}{2}=u,\ \mathrm{d}t=\mathrm{d}u,\ u<-\frac{\sqrt{5}}{2} \\ t^2+t-1=(t+\frac{1}{2})^2-\frac{5}{4}=u^2-\frac{5}{4} \end{array}\right] = \int \frac{\mathrm{d}u}{\sqrt{u^2-\frac{5}{4}}} = \ln\left|u+\sqrt{u^2-\tfrac{5}{4}}\right|+c_4$$

$$= \ln\left|t+\tfrac{1}{2}+\sqrt{t^2+t-1}\right|+c_4 = \ln\left|\frac{1}{x}+\frac{1}{2}+\frac{\sqrt{1+x-x^2}}{-x}\right|+c_4 = \ln\left|\frac{2+x-2\sqrt{1+x-x^2}}{2x}\right| + c_4$$

$$= \left[\ \frac{2+x-2\sqrt{1+x-x^2}}{2x}\,\frac{2+x+2\sqrt{1+x-x^2}}{2+x+2\sqrt{1+x-x^2}} = \frac{1}{2x}\,\frac{(2+x)^2-4(1+x-x^2)}{2+x+2\sqrt{1+x-x^2}} = \frac{1}{2x}\,\frac{5x^2}{2+x+2\sqrt{1+x-x^2}}\ \right]$$

$$= \ln\frac{5}{2} + \ln\left|\frac{x}{2+x+2\sqrt{1+x-x^2}}\right| + c_4 = \left[\ \ln\frac{5}{2}+c_4=c_5\ \right] = -\ln\left|\frac{2+x+2\sqrt{1+x-x^2}}{x}\right| + c_5,$$

for $x \in \left(\frac{1-\sqrt{5}}{2};\,0\right)$.

---

$$\int \frac{2x^2-x+1}{x\sqrt{1+x-x^2}}\,dx = \left[ \begin{array}{c} x^2-x-1=(x-\frac{1}{2})^2-\frac{5}{4}=\left(x-\frac{1}{2}-\frac{\sqrt{5}}{2}\right)\left(x-\frac{1}{2}+\frac{\sqrt{5}}{2}\right)<0 \\ x\in\left(\frac{1}{2}-\frac{\sqrt{5}}{2};\ \frac{1}{2}+\frac{\sqrt{5}}{2}\right),\ x\neq 0\ \Rightarrow\ x\in\left(\frac{1}{2}-\frac{\sqrt{5}}{2};\ 0\right)\cup\left(0;\ \frac{1}{2}+\frac{\sqrt{5}}{2}\right) \end{array} \right]$$

$$= \int\frac{(2x-1)\,dx}{\sqrt{1+x-x^2}} + \int\frac{dx}{x\sqrt{1+x-x^2}} = -2\sqrt{1+x-x^2} - \ln\left|\frac{2+x+2\sqrt{1+x-x^2}}{x}\right| + c,$$

for $x\in\left(\frac{1}{2}-\frac{\sqrt{5}}{2};\ 0\right)\cup\left(0;\ \frac{1}{2}+\frac{\sqrt{5}}{2}\right)$.

$$\int\frac{(2x-1)\,dx}{\sqrt{1+x-x^2}} = \left[ \begin{array}{c} 1+x-x^2=s \\ (1-2x)\,dx=ds \end{array} \right] = \int\frac{-ds}{\sqrt{s}} = -\int s^{-\frac{1}{2}}\,ds = -\frac{s^{\frac{1}{2}}}{\frac{1}{2}} + c_1 = -2\sqrt{s}+c_1$$

$= -2\sqrt{1+x-x^2}+c_1$, for $x\in\left(\frac{1}{2}-\frac{\sqrt{5}}{2};\ \frac{1}{2}+\frac{\sqrt{5}}{2}\right)$.

$$\int\frac{dx}{x\sqrt{1+x-x^2}} = \left[ \begin{array}{c} \boxed{\text{2nd ES}}\ \sqrt{-x^2+x+1}=xt+1,\ t=\frac{\sqrt{1+x-x^2}-1}{x},\ t\in\left(-\frac{1}{2}-\frac{\sqrt{5}}{2};\ -\frac{1}{2}+\frac{\sqrt{5}}{2}\right),\ t\neq 0 \\ -x^2+x+1=x^2t^2+2tx+1,\ x-2tx=x^2t^2+x^2,\ 1-2t=xt^2+x,\ x=\frac{1-2t}{t^2+1} \\ dx=\frac{-2(t^2+1)-2t(1-2t)}{(t^2+1)^2}\,dt=\frac{-2t^2-2-2t+4t^2}{(t^2+1)^2}\,dt=\frac{2t^2-2t-2}{(t^2+1)^2}\,dt=\frac{2(t^2-t-1)\,dt}{(t^2+1)^2} \\ \sqrt{1+x-x^2}=xt+1=\frac{1-2t}{t^2+1}t+1=\frac{t-2t^2+t^2+1}{t^2+1}=\frac{-t^2+t+1}{t^2+1}-\frac{t^2-t-1}{t^2+1} \end{array} \right]$$

$$= \int\frac{\frac{2(t^2-t-1)\,dt}{(t^2+1)^2}}{-\frac{1-2t}{t^2+1}\,\frac{t^2-t-1}{t^2+1}} = \int\frac{2\,dt}{2t-1} = \ln|2t-1|+c_2 = \ln\left|\frac{2\sqrt{1+x-x^2}-2-x}{x}\right|+c_2$$

$$= \left[ \frac{-2-x+\sqrt{1+x-x^2}}{x}\,\frac{2+x+2\sqrt{1+x-x^2}}{2+x+2\sqrt{1+x-x^2}} = \frac{1}{x}\,\frac{(2+x)^2-4(1+x-x^2)}{2+x+2\sqrt{1+x-x^2}} = \frac{1}{x}\,\frac{5x^2}{2+x+2\sqrt{1+x-x^2}} \right]$$

$$= \ln 5 + \ln\left|\frac{x}{2+x+2\sqrt{1+x-x^2}}\right| + c_2 = \left[\ \ln 5 + c_2 = c_3\ \right] = -\ln\left|\frac{2+x+2\sqrt{1+x-x^2}}{x}\right| + c_3,$$

for $x\in\left(\frac{1}{2}-\frac{\sqrt{5}}{2};\ 0\right)\cup\left(0;\ \frac{1}{2}+\frac{\sqrt{5}}{2}\right)$.

---

$\int\frac{\sqrt{x^2-x}}{x}\,dx = \frac{1}{2}\ln\left|2x-1-2\sqrt{x^2-x}\right| + \sqrt{x^2-x}+c$, for $x\in(-\infty\,;\,0)\cup\langle 1\,;\,\infty)$.

$$\int\frac{\sqrt{x^2-x}}{x}\,dx = \left[ \begin{array}{c} x>0,\ x=\sqrt{x^2},\ x^2-x=x(x-1)\geq 0,\ x\in(-\infty\,;\,0)\cup\langle 1\,;\,\infty)\ \Rightarrow\ x\in\langle 1\,;\,\infty) \\ t=\sqrt{\frac{x-1}{x}}=\frac{\sqrt{x-1}}{\sqrt{x}},\ t\in(0\,;\,1),\ \frac{x-1}{x}=t^2,\ x=\frac{1}{1-t^2},\ dx=\frac{-(-2t)\,dt}{(1-t^2)^2}=\frac{2t\,dt}{(1-t^2)^2} \\ 1\pm t=1\pm\frac{\sqrt{x-1}}{\sqrt{x}}=\frac{\sqrt{x}\pm\sqrt{x-1}}{\sqrt{x}},\ \frac{t}{1-t^2}=xt=x\sqrt{\frac{x-1}{x}}=\sqrt{x}\sqrt{x-1}=\sqrt{x^2-x} \\ \frac{1-t}{1+t}=\frac{\sqrt{x}-\sqrt{x-1}}{\sqrt{x}+\sqrt{x-1}}=\frac{\sqrt{x}-\sqrt{x-1}}{\sqrt{x}+\sqrt{x-1}}\,\frac{\sqrt{x}-\sqrt{x-1}}{\sqrt{x}-\sqrt{x-1}}=\frac{(\sqrt{x}-\sqrt{x-1})^2}{x-(x-1)}=(\sqrt{x}-\sqrt{x-1})^2>0 \end{array} \right]$$

$$= \int\frac{\sqrt{x^2-x}}{\sqrt{x^2}}\,dx = \int\sqrt{\frac{x^2-x}{x^2}}\,dx = \int\sqrt{\frac{x-1}{x}}\,dx = \int\frac{t\cdot 2t\,dt}{(1-t^2)^2} = \int\frac{2t^2\,dt}{(1-t^2)^2}$$

$$= \left[ \begin{array}{c} \frac{2t^2}{(1-t^2)^2}=\frac{A}{t-1}+\frac{B}{(t-1)^2}+\frac{C}{t+1}+\frac{D}{(t+1)^2} \\ A=\frac{1}{2},\ B=\frac{1}{2},\ C=-\frac{1}{2},\ D=\frac{1}{2} \end{array} \right] = \frac{1}{2}\int\left[\frac{1}{t-1}+\frac{1}{(t-1)^2}-\frac{1}{t+1}+\frac{1}{(t+1)^2}\right]dt$$

$$= \frac{1}{2}\left[\ln|t-1|+\frac{(t-1)^{-1}}{-1}-\ln|t+1|+\frac{(t+1)^{-1}}{-1}\right]+c = \frac{1}{2}\left[\ln\left|\frac{t-1}{t+1}\right|+\frac{1}{1-t}-\frac{1}{1+t}\right]+c$$

$$= \frac{1}{2}\left[\ln\frac{1-t}{1+t}+\frac{1+t-(1-t)}{1-t^2}\right]+c = \frac{1}{2}\ln\frac{1-t}{1+t}+\frac{t}{1-t^2}+c = \frac{1}{2}\ln(\sqrt{x}-\sqrt{x-1})^2+\sqrt{x^2-x}+c,$$

$= \frac{1}{2}\ln\left(2x-1-2\sqrt{x^2-x}\right)+\sqrt{x^2-x}+c$, for $x\in\langle 1\,;\,\infty)$.

$$\int\frac{\sqrt{x^2-x}}{x}\,dx = \int\frac{\sqrt{x^2-x}}{-\sqrt{x^2}}\,dx = -\int\sqrt{\frac{x-1}{x}}\,dx = -\int\frac{2t^2\,dt}{(1-t^2)^2} = -\frac{1}{2}\left[\ln\left|\frac{t-1}{t+1}\right|+\frac{1}{1-t}-\frac{1}{1+t}\right]+c$$

$$= \left[ \begin{array}{c} x<0,\ x=-\sqrt{x^2},\ t=\sqrt{\frac{x-1}{x}}=\frac{\sqrt{1-x}}{\sqrt{-x}},\ t\in(0\,;\,1),\ \frac{x-1}{x}=t^2,\ x=\frac{1}{1-t^2},\ dx=\frac{2t\,dt}{(1-t^2)^2} \\ t\pm 1=\frac{\sqrt{1-x}}{\sqrt{-x}}\pm 1=\frac{\sqrt{1-x}\pm\sqrt{-x}}{\sqrt{-x}},\ \frac{t}{1-t^2}=xt=-\sqrt{x^2}\sqrt{\frac{x-1}{x}}=-\sqrt{x(x-1)}=-\sqrt{x^2-x} \\ \frac{t+1}{t-1}=\frac{\sqrt{1-x}+\sqrt{-x}}{\sqrt{1-x}-\sqrt{-x}}=\frac{\sqrt{1-x}+\sqrt{-x}}{\sqrt{1-x}-\sqrt{-x}}\,\frac{\sqrt{1-x}+\sqrt{-x}}{\sqrt{1-x}+\sqrt{-x}}=\frac{(\sqrt{1-x}+\sqrt{-x})^2}{1-x-(-x)}=(\sqrt{1-x}+\sqrt{-x})^2>0 \end{array} \right]$$

$$= -\frac{1}{2}\ln\frac{t-1}{t+1}-\frac{t}{1-t^2}+c = \frac{1}{2}\ln\frac{t+1}{t-1}-\frac{t}{1-t^2}+c = \frac{1}{2}\ln(\sqrt{1-x}+\sqrt{-x})^2+\sqrt{x^2-x}+c$$

$= \frac{1}{2}\ln\left(1-2x+2\sqrt{-x(1-x)}\right)+\sqrt{x^2-x}+c$, for $x\in(-\infty\,;\,0)$.

---

$$\int \frac{x^5\,\mathrm{d}x}{\sqrt{x^3+1}} = \frac{1}{3}\int \frac{x^3\cdot 3x^2\,\mathrm{d}x}{\sqrt{x^3+1}} = \left[\begin{array}{c} x^3+1=t,\ x>-1 \\ 3x^2\,\mathrm{d}x=\mathrm{d}t,\ t>0 \end{array}\right] = \frac{1}{3}\int \frac{(t-1)\,\mathrm{d}t}{\sqrt{t}} = \frac{1}{3}\int \left[\sqrt{t}-\frac{1}{\sqrt{t}}\right]\mathrm{d}t$$

$$= \frac{1}{3}\int \left[t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right]\mathrm{d}t = \frac{1}{3}\left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}} - \frac{t^{\frac{1}{2}}}{\frac{1}{2}}\right] + c = \frac{2\sqrt{t^3}}{9} - \frac{2\sqrt{t}}{3} + c = \frac{2\sqrt{(x^3+1)^3}}{9} - \frac{2\sqrt{x^3+1}}{3} + c,$$

for $x\in(-1\,;\,\infty)$.

---

$$\int \frac{\mathrm{d}x}{\sqrt{5+4\,\mathrm{e}^x}} = \left[\begin{array}{c} \mathrm{e}^x=t,\ x\in R,\ t>0 \\ x=\ln t,\ \mathrm{d}x=\frac{\mathrm{d}t}{t} \end{array}\right] = \int \frac{\mathrm{d}t}{t\sqrt{5+4t}} = \int \frac{4\,\mathrm{d}t}{4t\sqrt{5+4t}} = \left[\begin{array}{c} 5+4t=u^2,\ t>0 \\ 4\,\mathrm{d}t=2u\,\mathrm{d}u,\ u>\sqrt{5} \end{array}\right]$$

$$= \int \frac{2u\,\mathrm{d}u}{(u^2-5)\sqrt{u^2}} = \int \frac{2\,\mathrm{d}u}{u^2-5} = \frac{2}{2\sqrt{5}}\ln\left|\frac{u-\sqrt{5}}{u+\sqrt{5}}\right| + c = \frac{1}{\sqrt{5}}\ln\left|\frac{\sqrt{5+4t}-\sqrt{5}}{\sqrt{5+4t}+\sqrt{5}}\right| + c$$

$$= \frac{1}{\sqrt{5}}\ln\left|\frac{\sqrt{5+4\,\mathrm{e}^x}-\sqrt{5}}{\sqrt{5+4\,\mathrm{e}^x}+\sqrt{5}}\right| + c,$$ for $x\in R$.

---

$$\int \frac{\sqrt{1-x^2}}{x^2}\,\mathrm{d}x = \left[\begin{array}{l|l} u=\sqrt{1-x^2} & u'=\frac{1}{2}\left(1-x^2\right)^{-\frac{1}{2}}(-2x)=\frac{-x}{\sqrt{1-x^2}} \\ v'=\frac{1}{x^2}=x^{-2} & v=\frac{x^{-1}}{-1}=-\frac{1}{x} \end{array}\right] = -\frac{\sqrt{1-x^2}}{x} - \int \frac{-x}{\sqrt{1-x^2}}\,\frac{-\mathrm{d}x}{x}$$

$$= -\frac{\sqrt{1-x^2}}{x} - \int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = -\frac{\sqrt{1-x^2}}{x} - \arcsin x + c,$$ for $x\in\langle -1\,;\,1\rangle - \{0\}$.

---

$$\int \frac{\sqrt{1-x^2}}{x^2}\,\mathrm{d}x = \left[\begin{array}{c} x=\sin t,\ t=\arcsin x,\ \mathrm{d}x=\cos t\,\mathrm{d}t,\ x\in\langle -1\,;\,1\rangle,\ t\in\left\langle -\frac{\pi}{2}\,;\,\frac{\pi}{2}\right\rangle \\ \sqrt{1-x^2}=\sqrt{1-\sin^2 t}=\sqrt{\cos^2 t}=|\cos t|=\cos t \end{array}\right] = \int \frac{\cos t\cdot\cos t\,\mathrm{d}t}{\sin^2 t}$$

$$= \int \frac{1-\sin^2 t\,\mathrm{d}t}{\sin^2 t} = \int \left[\frac{1}{\sin^2 t} - 1\right]\mathrm{d}t = -\operatorname{cotg} t - t + c = c - \frac{\cos t}{\sin t} - t$$

$$= c - \frac{\sqrt{1-x^2}}{x} - \arcsin x,$$ for $x\in\langle -1\,;\,1\rangle - \{0\}$.

---

$$\int \sqrt{\left(\frac{1-x}{1+x}\right)^3}\,\mathrm{d}x = \left[\begin{array}{c} \frac{1-x}{1+x}>0 \Leftrightarrow (1-x)(1+x)=1-x^2>0,\ x\in(-1\,;\,1\rangle,\ t^2=\frac{1-x}{1+x},\ t\in\langle 0\,;\,\infty) \\ 1-x=t^2(1+x),\ x=\frac{1-t^2}{1+t^2},\ \mathrm{d}x=\frac{-2t(1+t^2)-2t(1-t^2)}{(1+t^2)^2}\,\mathrm{d}t=\frac{-4t\,\mathrm{d}t}{(1+t^2)^2},\ t^2+1=\frac{2}{1+x} \end{array}\right]$$

$$= \int \frac{-4t\cdot t^3\,\mathrm{d}t}{(1+t^2)^2} = \left[\frac{t^4}{(1+t^2)^2} = \frac{t^4+2t^2+1-2t^2-2+1}{(1+t^2)^2} = \frac{(t^2+1)^2-2(t^2+1)+1}{(1+t^2)^2} = 1-\frac{2}{1+t^2}+\frac{1}{(1+t^2)^2}\right]$$

$$= -4\int \left[1-\frac{2}{1+t^2}+\frac{1}{(1+t^2)^2}\right]\mathrm{d}t = \left[\text{p. 21}: \int \frac{\mathrm{d}t}{(1+t^2)^2} = \frac{1}{2}\operatorname{arctg} t + \frac{t}{2(t^2+1)},\ t\in R\right]$$

$$= -4\left[t - 2\operatorname{arctg} t + \frac{1}{2}\operatorname{arctg} t + \frac{1}{2}\frac{t}{t^2+1}\right] + c = 6\operatorname{arctg} t - 4t - \frac{2t}{t^2+1} + c$$

$$= 6\operatorname{arctg}\sqrt{\frac{1-x}{1+x}} - 4\sqrt{\frac{1-x}{1+x}} - 2\sqrt{\frac{1-x}{1+x}}\,\frac{1+x}{2} + c$$

$$= 6\operatorname{arctg}\sqrt{\frac{1-x}{1+x}} - 4\sqrt{\frac{1-x}{1+x}} - \sqrt{1-x^2} + c,$$ for $x\in(-1\,;\,1\rangle$.

---

$$\int \sqrt{\left(\frac{1+x}{1-x}\right)^3}\,\mathrm{d}x = \left[\begin{array}{c} x=-t,\ x\in\langle -1\,;\,1) \\ \mathrm{d}x=-\,\mathrm{d}t,\ t\in(-1\,;\,1\rangle \end{array}\right] = -\int \sqrt{\left(\frac{1-t}{1+t}\right)^3}\,\mathrm{d}t$$

$$= -\left[6\operatorname{arctg}\sqrt{\frac{1-t}{1+t}} - 4\sqrt{\frac{1-t}{1+t}} - \sqrt{1-t^2}\right] + c$$

$$= -6\operatorname{arctg}\sqrt{\frac{1+x}{1-x}} + 4\sqrt{\frac{1+x}{1-x}} + \sqrt{1-x^2} + c,$$ for $x\in\langle -1\,;\,1)$.

---

$$\int\sqrt{\left(\tfrac{1+x}{1-x}\right)^3}\,\mathrm{d}x = \left[\begin{array}{c} \frac{1+x}{1-x}>0 \Leftrightarrow (1-x)(1+x)=1-x^2>0,\ x\in\langle -1\,;\,1),\ t^2=\frac{1+x}{1-x},\ t\in\langle 0\,;\,\infty) \\ 1+x=t^2(1-x),\ x=\frac{t^2-1}{t^2+1},\ \mathrm{d}x=\frac{2t(t^2+1)-2t(t^2-1)}{(t^2+1)^2}\,\mathrm{d}t=\frac{4t\,\mathrm{d}t}{(t^2+1)^2},\ t^2+1=\frac{2}{1-x}\end{array}\right]$$
$$= \int\frac{4t\cdot t^3\,\mathrm{d}t}{(t^2+1)^2} = \left[\ \frac{t^4}{(t^2+1)^2}=\frac{t^4+2t^2+1-2t^2-2+1}{(t^2+1)^2}=\frac{(t^2+1)^2-2(t^2+1)+1}{(t^2+1)^2}=1-\frac{2}{1+t^2}+\frac{1}{(t^2+1)^2}\ \right]$$
$$= 4\int\left[1-\frac{2}{1+t^2}+\frac{1}{(t^2+1)^2}\right]\mathrm{d}t = \left[\ \text{p. } 21: \int\frac{\mathrm{d}t}{(1+t^2)^2}=\frac{1}{2}\operatorname{arctg} t+\frac{t}{2(t^2+1)},\ t\in R\ \right]$$
$$= 4\left[t-2\operatorname{arctg} t+\frac{1}{2}\operatorname{arctg} t+\frac{1}{2}\frac{t}{t^2+1}\right]+c = -6\operatorname{arctg} t+4t+\frac{2t}{t^2+1}+c$$
$$= -6\operatorname{arctg}\sqrt{\frac{1+x}{1-x}}+4\sqrt{\frac{1+x}{1-x}}+2\sqrt{\frac{1+x}{1-x}}\,\frac{1-x}{2}+c$$
$$= -6\operatorname{arctg}\sqrt{\frac{1+x}{1-x}}+4\sqrt{\frac{1+x}{1-x}}+\sqrt{1-x^2}+c, \text{ for } x\in\langle -1\,;\,1).$$

---

$$\int x^2\ln\sqrt{1-x}\,\mathrm{d}x = \int\frac{x^2}{2}\ln(1-x)\,\mathrm{d}x = \left[\begin{array}{l|l} u=\ln(1-x) & u'=\frac{-1}{1-x}=\frac{1}{x-1} \\ v'=\frac{x^2}{2} & v=\frac{x^3}{2\cdot 3}=\frac{x^3}{6}\end{array}\right]$$
$$= \frac{x^3\ln(1-x)}{6}-\int\frac{x^3}{6}\frac{\mathrm{d}x}{x-1} = \frac{x^3\ln(1-x)}{6}-\frac{1}{6}\int\frac{(x^3-x^2)+(x^2-x)+(x-1)+1}{x-1}\,\mathrm{d}x$$
$$= \frac{x^3\ln(1-x)}{6}-\frac{1}{6}\int\left[x^2+x+1+\frac{1}{x-1}\right]\mathrm{d}x = \frac{x^3\ln(1-x)}{6}-\frac{1}{6}\left[\frac{x^3}{3}+\frac{x^2}{2}+x+\ln|x-1|\right]+c$$
$$= \frac{x^3\ln(1-x)}{6}-\frac{x^3}{18}-\frac{x^2}{12}-\frac{x}{6}-\frac{\ln(1-x)}{6}+c = \frac{x^3-1}{6}\ln(1-x)-\frac{x^3}{18}-\frac{x^2}{12}-\frac{x}{6}+c, \text{ for } x<-1.$$

---

$$\int\frac{\mathrm{d}x}{x^6(1+x^2)} = \left[\begin{array}{c}\frac{1}{x^6(1+x^2)}=\frac{1}{t^3(1+t)}=\frac{A}{t}+\frac{B}{t^2}+\frac{C}{t^3}+\frac{D}{1+t}=\frac{A}{x^2}+\frac{B}{x^4}+\frac{C}{x^6}+\frac{D}{1+x^2} \\ x^2=t,\ A=1,\ B=-1,\ C=1,\ D=-1\end{array}\right]$$
$$= \int\left[\frac{1}{x^2}-\frac{1}{x^4}+\frac{1}{x^6}-\frac{1}{1+x^2}\right]\mathrm{d}x = \int\left[x^{-2}-x^{-4}+x^{-6}-\frac{1}{1+x^2}\right]\mathrm{d}x$$
$$= \frac{x^{-1}}{-1}-\frac{x^{-3}}{-3}+\frac{x^{-5}}{-5}-\operatorname{arctg} x+c = -\frac{1}{x}+\frac{1}{3x^3}-\frac{1}{5x^5}-\operatorname{arctg} x+c, \text{ for } x\in R-\{0\}.$$

---

$$\int\sqrt{\frac{1-x}{1+x}}\,\mathrm{d}x = \left[\begin{array}{c}\frac{1-x}{1+x}>0\Leftrightarrow(1-x)(1+x)=1-x^2>0,\ x\in(-1\,;\,1\rangle,\ t=\sqrt{\frac{1-x}{1+x}},\ t^2=\frac{1-x}{1+x}, t\in\langle 0\,;\,\infty) \\ 1-x=t^2(1+x),\ x=\frac{1-t^2}{1+t^2},\ \mathrm{d}x=\frac{-2t(1+t^2)-2t(1-t^2)}{(1+t^2)^2}\,\mathrm{d}t=\frac{-4t\,\mathrm{d}t}{(1+t^2)^2},\ t^2+1=\frac{2}{1+x}\end{array}\right]$$
$$= \int\frac{-4t\cdot t\,\mathrm{d}t}{(1+t^2)^2} = -4\int\frac{t^2+1-1}{(1+t^2)^2}\,\mathrm{d}t = -4\int\left[\frac{1}{1+t^2}-\frac{1}{(1+t^2)^2}\right]\mathrm{d}t$$
$$= \left[\ \text{p. } 21: \int\frac{\mathrm{d}t}{(1+t^2)^2}=\frac{1}{2}\operatorname{arctg} t+\frac{t}{2(t^2+1)},\ t\in R\ \right] = -4\left[\operatorname{arctg} t-\frac{1}{2}\operatorname{arctg} t-\frac{1}{2}\frac{t}{t^2+1}\right]+c$$
$$= -2\operatorname{arctg} t+\frac{2t}{t^2+1}+c = \left[\ \frac{2t}{t^2+1}=2\sqrt{\frac{1-x}{1+x}}\,\frac{1+x}{2}=\sqrt{\frac{(1-x)(1+x)^2}{1+x}}=\sqrt{(1-x)(1+x)}=\sqrt{1-x^2}\ \right]$$
$$= \sqrt{1-x^2}-2\operatorname{arctg}\sqrt{\frac{1-x}{1+x}}+c, \text{ for } x\in(-1\,;\,1\rangle.$$

---

$$\int\sqrt{\frac{1+x}{1-x}}\,\mathrm{d}x = \left[\begin{array}{c} x=-t,\ x\in\langle -1\,;\,1) \\ \mathrm{d}x=-\,\mathrm{d}t,\ t\in(-1\,;\,1\rangle\end{array}\right] = -\int\sqrt{\frac{1-t}{1+t}}\,\mathrm{d}t = -\left[\sqrt{1-t^2}-2\operatorname{arctg}\sqrt{\frac{1-t}{1+t}}\right]+c,$$
$$= -\sqrt{1-x^2}+2\operatorname{arctg}\sqrt{\frac{1+x}{1-x}}+c, \text{ for } x\in\langle -1\,;\,1).$$

---

$$\int\sqrt{\frac{1+x}{1-x}}\,\mathrm{d}x = \left[\begin{array}{c} \frac{1+x}{1-x}>0 \Leftrightarrow (1-x)(1+x)=1-x^2>0,\ x\in\langle-1\,;\,1)\,,\ t=\sqrt{\frac{1+x}{1-x}},\ t^2=\frac{1+x}{1-x}, t\in\langle 0\,;\,\infty) \\ 1+x=t^2(1-x),\ x=\frac{t^2-1}{t^2+1},\ \mathrm{d}x=\frac{2t(t^2+1)-2t(t^2-1)}{(t^2+1)^2}\,\mathrm{d}t=\frac{4t\,\mathrm{d}t}{(t^2+1)^2},\ t^2+1=\frac{2}{1-x} \end{array}\right]$$
$$= \int\frac{4t\cdot t\,\mathrm{d}t}{(1+t^2)^2} = 4\int\frac{t^2+1-1}{(1+t^2)^2}\,\mathrm{d}t = 4\int\left[\frac{1}{1+t^2}-\frac{1}{(1+t^2)^2}\right]\mathrm{d}t$$
$$= \left[\text{p. } 21: \int\frac{\mathrm{d}t}{(1+t^2)^2}=\frac{1}{2}\operatorname{arctg}t+\frac{t}{2(t^2+1)},\ t\in R\right] = 4\left[\operatorname{arctg}t-\frac{1}{2}\operatorname{arctg}t-\frac{1}{2}\frac{t}{t^2+1}\right]+c$$
$$= 2\operatorname{arctg}t-\frac{2t}{t^2+1}+c = \left[\frac{2t}{t^2+1}=2\sqrt{\frac{1+x}{1-x}}\,\frac{1-x}{2}=\sqrt{\frac{(1+x)(1-x)^2}{1-x}}=\sqrt{(1+x)(1-x)}=\sqrt{1-x^2}\right]$$
$$= 2\operatorname{arctg}\sqrt{\frac{1+x}{1-x}}-\sqrt{1-x^2}+c, \text{ for } x\in\langle-1\,;\,1).$$

---

$$\int\sqrt{\frac{x-1}{x+1}}\,\mathrm{d}x = \left[\begin{array}{c} \frac{x-1}{x+1}>0 \Leftrightarrow (x-1)(x+1)=x^2-1>0,\ x\in(-\infty\,;\,-1)\cup\langle 1\,;\,\infty)\,,\ t=\sqrt{\frac{x-1}{x+1}}, t^2=\frac{x-1}{x+1} \\ x-1=t^2(x+1),\ x=\frac{1+t^2}{1-t^2},\ \mathrm{d}x=\frac{2t(1-t^2)+2t(1+t^2)}{(1-t^2)^2}\,\mathrm{d}t=\frac{4t\,\mathrm{d}t}{(t^2-1)^2},\ t\in\langle 0\,;\,\infty)-\{1\} \end{array}\right]$$
$$= \int\frac{4t\cdot t\,\mathrm{d}t}{(t^2-1)^2} = \left[\begin{array}{c} \frac{4t^2}{(t^2-1)^2}=\frac{A}{t-1}+\frac{B}{(t-1)^2}+\frac{C}{t+1}+\frac{D}{(t+1)^2} \\ A=1,\ B=1,\ C=-1,\ D=1 \end{array}\right] = \int\left[\frac{1}{t-1}+\frac{1}{(t-1)^2}-\frac{1}{t+1}+\frac{1}{(t+1)^2}\right]\mathrm{d}t$$
$$= \ln|t-1|+\int(t-1)^{-2}\,\mathrm{d}t-\ln|t+1|+\int(t+1)^{-2}\,\mathrm{d}t = \ln\left|\frac{t-1}{t+1}\right|+\frac{(t-1)^{-1}}{-1}+\frac{(t+1)^{-1}}{-1}+c$$
$$= \ln\left|\frac{t-1}{t+1}\right|-\frac{1}{t-1}-\frac{1}{t+1}+c = \ln|t-1|-\ln|t+1|-\frac{2t}{t^2-1}+c$$
$$= \left[t^2-1=\frac{x-1}{x+1}-1=\frac{-2}{x+1},\ -\frac{2t}{t^2-1}=(x+1)\sqrt{\frac{x-1}{x+1}}=\sqrt{(x+1)^2\frac{x-1}{x+1}}=\sqrt{(x+1)(x-1)}=\sqrt{x^2-1}\right]$$
$$= \ln\left|\sqrt{\frac{x-1}{x+1}}-1\right|-\ln\left|\sqrt{\frac{x-1}{x+1}}+1\right|+\sqrt{x^2-1}+c, \text{ for } x\in(-\infty\,;\,-1)\cup\langle 1\,;\,\infty).$$

---

$$\int\sqrt{\frac{x+1}{x-1}}\,\mathrm{d}x = \left[\begin{array}{c} \frac{x+1}{x-1}>0 \Leftrightarrow (x-1)(x+1)=x^2-1>0,\ x\in(-\infty\,;\,-1\rangle\cup(1\,;\,\infty)\,,\ t=\sqrt{\frac{x+1}{x-1}}, t^2=\frac{x+1}{x-1} \\ x+1=t^2(x-1),\ x=\frac{t^2+1}{t^2-1},\ \mathrm{d}x=\frac{2t(t^2-1)-2t(t^2+1)}{(t^2-1)^2}\,\mathrm{d}t=\frac{-4t\,\mathrm{d}t}{(t^2-1)^2},\ t\in\langle 0\,;\,\infty)-\{1\} \end{array}\right]$$
$$= \int\frac{-4t\cdot t\,\mathrm{d}t}{(t^2-1)^2} = \left[\begin{array}{c} \frac{-4t^2}{(t^2-1)^2}=\frac{A}{t+1}+\frac{B}{(t+1)^2}+\frac{C}{t-1}+\frac{D}{(t-1)^2} \\ A=1,\ B=-1,\ C=-1,\ D=-1 \end{array}\right] = \int\left[\frac{1}{t+1}-\frac{1}{(t+1)^2}-\frac{1}{t-1}-\frac{1}{(t-1)^2}\right]\mathrm{d}t$$
$$= \ln|t+1|-\int(t+1)^{-2}\,\mathrm{d}t-\ln|t-1|-\int(t-1)^{-2}\,\mathrm{d}t = \ln\left|\frac{t+1}{t-1}\right|-\frac{(t+1)^{-1}}{-1}-\frac{(t-1)^{-1}}{-1}+c$$
$$= \ln\left|\frac{t+1}{t-1}\right|+\frac{1}{t-1}+\frac{1}{t+1}+c = \ln|t+1|-\ln|t-1|+\frac{2t}{t^2-1}+c$$
$$= \left[t^2-1=\frac{x+1}{x-1}-1=\frac{2}{x-1},\ \frac{2t}{t^2-1}=(x-1)\sqrt{\frac{x+1}{x-1}}=\sqrt{(x-1)^2\frac{x+1}{x-1}}=\sqrt{(x-1)(x+1)}=\sqrt{x^2-1}\right]$$
$$= \ln\left|\sqrt{\frac{x+1}{x-1}}+1\right|-\ln\left|\sqrt{\frac{x+1}{x-1}}-1\right|+\sqrt{x^2-1}+c, \text{ for } x\in(-\infty\,;\,-1\rangle\cup(1\,;\,\infty).$$

---

$$\int\frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}}\,\mathrm{d}x = \int\frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}}\,\frac{1+\sqrt{1-x^2}}{1+\sqrt{1-x^2}}\,\mathrm{d}x = \int\frac{1+2\sqrt{1-x^2}+1-x^2}{1-(1-x^2)}\,\mathrm{d}x = \int\frac{2-x^2+2\sqrt{1-x^2}}{x^2}\,\mathrm{d}x$$
$$= 2\int x^{-2}\,\mathrm{d}x-\int\mathrm{d}x+2\int\frac{\sqrt{1-x^2}}{x^2}\,\mathrm{d}x = \left[\begin{array}{l|l} u=\sqrt{1-x^2} & u'=\frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x)=\frac{-x}{\sqrt{1-x^2}} \\ v'=\frac{1}{x^2}=x^{-2} & v=\frac{x^{-1}}{-1}=-\frac{1}{x} \end{array}\right]$$
$$= 2\frac{x^{-1}}{-1}-x+2\left[-\frac{\sqrt{1-x^2}}{x}-\int\frac{-x}{\sqrt{1-x^2}}\frac{-\,\mathrm{d}x}{x}\right] = -\frac{2}{x}-x-2\frac{\sqrt{1-x^2}}{x}-2\int\frac{\mathrm{d}x}{\sqrt{1-x^2}}$$
$$= -\frac{2+x^2+2\sqrt{1-x^2}}{x}-2\arcsin x+c, \text{ for } x\in\langle-1\,;\,1\rangle-\{0\}.$$

---

$$\int\frac{dx}{x\sqrt{1\pm x^3+x^6}} = -\frac13\ln\left|\frac{2\pm x^3+2\sqrt{1\pm x^3+x^6}}{x^3}\right| + c, \text{ for } x\in R-\{0\}.$$

$$\int\frac{dx}{x\sqrt{1\pm x^3+x^6}} = \left[\, x^6\pm x^3+1=(x^3\pm\tfrac12)^2+\tfrac34>0 \,\right] = \frac13\int\frac{3x^2\,dx}{x^3\sqrt{1\pm x^3+x^6}} = \left[\begin{array}{c} t=x^3,\ x>0\\ 3x^2\,dx=dt,\ t>0\end{array}\right]$$

$$=\frac13\int\frac{dt}{t\sqrt{1\pm t+t^2}} = \left[\begin{array}{l} t=\frac1u,\ \sqrt{1\pm t+t^2}=\sqrt{1\pm\frac1u+\frac1{u^2}}=\frac{\sqrt{u^2\pm u+1}}{|u|}=\frac{\sqrt{u^2\pm u+1}}{u},\ t>0\\ dt=\frac{-du}{u^2},\ \sqrt{u^2\pm u+1}=\sqrt{\frac1{t^2}\pm\frac1t+1}=\frac{\sqrt{1\pm t+t^2}}{|t|}=\frac{\sqrt{1\pm t+t^2}}{t},\ u>0\end{array}\right]$$

$$=\frac13\int\frac{u\cdot u}{\sqrt{u^2\pm u+1}}\,\frac{-du}{u^2} = -\frac13\int\frac{du}{\sqrt{(u\pm\frac12)^2+\frac34}} = -\frac13\ln\left|u\pm\frac12+\sqrt{u^2\pm u+1}\right|+c_1$$

$$=-\frac13\ln\left|\frac1t\pm\frac12+\frac{\sqrt{1\pm t+t^2}}{t}\right|+c_1 = -\frac13\ln\left|\frac{2\pm t+2\sqrt{1\pm t+t^2}}{2t}\right|+c_1 = \left[\,\tfrac13\ln2+c_1=c\,\right]$$

$$=-\frac13\ln\left|\frac{2\pm t+2\sqrt{1\pm t+t^2}}{t}\right|+\frac13\ln2+c_1 = -\frac13\ln\left|\frac{2\pm x^3+2\sqrt{1\pm x^3+x^6}}{x^3}\right|+c, \text{ for } x>0.$$

$$\int\frac{dx}{x\sqrt{1\pm x^3+x^6}} = \left[\begin{array}{c} t=x^3, x<0\\ 3x^2\,dx=dt, t<0\end{array}\right] = \frac13\int\frac{dt}{t\sqrt{1\pm t+t^2}} = \left[\begin{array}{l} t=\frac1u,\ \sqrt{1\pm t+t^2}=\frac{\sqrt{u^2\pm u+1}}{-u},\ t<0\\ dt=\frac{-du}{u^2}, \sqrt{u^2\pm u+1}=\frac{\sqrt{1\pm t+t^2}}{-t}, u<0\end{array}\right]$$

$$=\frac13\int\frac{-u\cdot u}{\sqrt{u^2\pm u+1}}\,\frac{-du}{u^2}=\frac13\int\frac{du}{\sqrt{(u\pm\frac12)^2+\frac34}}=\frac13\ln\left|u\pm\frac12+\sqrt{u^2\pm u+1}\right|+c_1$$

$$=\frac13\ln\left|\frac1t\pm\frac12-\frac{\sqrt{1\pm t+t^2}}{t}\right|+c_1=\frac13\ln\left|\frac{2\pm t-2\sqrt{1\pm t+t^2}}{2t}\right|+c_1$$

$$=\left[\,\frac{2\pm t-2\sqrt{1\pm t+t^2}}{2t}\,\frac{2\pm t+2\sqrt{1\pm t+t^2}}{2\pm t+2\sqrt{1\pm t+t^2}}=\frac1{2t}\,\frac{(2\pm t)^2-4(1\pm t+t^2)}{2\pm t+2\sqrt{1\pm t+t^2}}=\frac1{2t}\,\frac{-3t^2}{2\pm t+2\sqrt{1\pm t+t^2}}\,\right]$$

$$=c_1-\frac13\ln\left|\frac23\,\frac{2\pm t+2\sqrt{1\pm t+t^2}}{t}\right|=\left[\,c_1-\tfrac13\ln\tfrac23=c\,\right]=-\frac13\ln\left|\frac{2\pm x^3+2\sqrt{1\pm x^3+x^6}}{x^3}\right|+c, \text{ for } x<0.$$

---

$$\int\sqrt{\frac{x}{1-x\sqrt x}}\,dx=\left[\begin{array}{c}\sqrt x=t,\ x\ge0,\ 1-x\sqrt x=1-\sqrt{x^3}>0,\ x<1\\ x=t^2,\ dx=2t\,dt,\ x\in\langle0;1),\ t\in\langle0;1)\end{array}\right]=\int\sqrt{\frac{t^2}{1-t^3}}\,2t\,dt=\int\frac{2t^2\,dt}{\sqrt{1-t^3}}$$

$$=\left[\begin{array}{c}1-t^3=u\\-3t^2\,dt=du\end{array}\right]=-\frac23\int\frac{du}{\sqrt u}=-\frac23\int u^{-\frac12}\,du=-\frac23\,\frac{u^{\frac12}}{\frac12}+c=-\frac43\sqrt u+c$$

$$=-\frac43\sqrt{1-t^3}+c=-\frac43\sqrt{1-x\sqrt x}+c, \text{ for } x\in\langle0;1).$$

---

$$\int\frac{1-x}{x\sqrt{x-x^2}}\,dx=\left[\begin{array}{l}\boxed{\text{3rd ES}}\ t=\sqrt{\frac{1-x}{x}},\ t^2=\frac{1-x}{x},\ t^2x=1-x,\ x=\frac1{1+t^2},\ dx=\frac{-1\cdot2t}{(1+t^2)^2}\,dt=\frac{-2t\,dt}{(1+t^2)^2}\\ x-x^2=x(1-x)>0,\ x\in(0;1),\ t\in(0;\infty),\ t=\sqrt{\frac{1-x}{x}}=\sqrt{\frac{x-x^2}{x^2}}=\frac{\sqrt{x-x^2}}{x}\\ 1-x=t^2x=\frac{t^2}{1+t^2},\ \sqrt{x-x^2}=xt=\frac{t}{1+t^2},\ \frac{1-x}{x\sqrt{x-x^2}}=\frac{1+t^2}{1}\,\frac{t^2}{1+t^2}\,\frac{1+t^2}{t}=t(1+t^2)\end{array}\right]$$

$$=\int t(1+t^2)\frac{-2t\,dt}{(1+t^2)^2}=-2\int\frac{t^2\,dt}{1+t^2}=-2\int\frac{1+t^2-1}{1+t^2}\,dt=-2\int\left[1-\frac1{1+t^2}\right]dt$$

$$=-2(t-\operatorname{arctg}t)+c=2\operatorname{arctg}\sqrt{\frac{1-x}{x}}-2\sqrt{\frac{1-x}{x}}+c, \text{ for } x\in(0;1).$$

---

$$\int\frac{1-x}{x-\sqrt{x-x^2}}\,dx=\left[\begin{array}{l}\boxed{\text{3rd ES}}\ t=\sqrt{\frac{1-x}{x}},\ t^2=\frac{1-x}{x},\ t^2x=1-x,\ x=\frac1{1+t^2},\ dx=\frac{-1\cdot2t}{(1+t^2)^2}\,dt=\frac{-2t\,dt}{(1+t^2)^2}\\ x-x^2=x(1-x)>0,\ x\in(0;1),\ t\in(0;\infty),\ t=\sqrt{\frac{1-x}{x}}=\sqrt{\frac{x-x^2}{x^2}}=\frac{\sqrt{x-x^2}}{x}\\ 1-x=t^2x=\frac{t^2}{1+t^2},\sqrt{x-x^2}=xt=\frac{t}{1+t^2},\ x\pm\sqrt{x-x^2}=\frac1{1+t^2}\pm\frac t{1+t^2}=\frac{1\pm t}{1+t^2}\end{array}\right]$$

$$=\int\frac{t^2}{1+t^2}\,\frac{1+t^2}{1-t}\,\frac{-2t\,dt}{(1+t^2)^2}=\int\frac{2t^3\,dt}{(t-1)(1+t^2)^2}=\left[\begin{array}{l}\frac{2t^3}{(t-1)(1+t^2)^2}=\frac{A}{t-1}+\frac{Bt+C}{1+t^2}+\frac{Dt+E}{(1+t^2)^2}\\ A=\frac12, B=-\frac12, C=\frac32, D=1, E=-1\end{array}\right]$$

$$=\int\left[\frac{\frac12}{t-1}+\frac{\frac32-\frac t2}{1+t^2}+\frac{t-1}{(1+t^2)^2}\right]dt=\frac12\int\frac{dt}{t-1}+\frac32\int\frac{dt}{1+t^2}-\frac12\int\frac{t\,dt}{1+t^2}+\int\frac{t\,dt}{(1+t^2)^2}-\int\frac{dt}{(1+t^2)^2}$$

$$= \left[ \begin{array}{l} 1+t^2=u,\ t\in(0\,;\,\infty) \\ 2t\,\mathrm{d}t=\mathrm{d}u,\ u\in(1\,;\,\infty) \end{array} \middle|\ \text{p. 21}: \int\frac{\mathrm{d}t}{(1+t^2)^2}=\tfrac{1}{2}\operatorname{arctg}t+\tfrac{t}{2(1+t^2)},\ t\in R \right]$$

$$= \tfrac{1}{2}\ln|t-1|+\tfrac{3}{2}\operatorname{arctg}t-\tfrac{1}{4}\int\frac{\mathrm{d}u}{u}+\tfrac{1}{2}\int\frac{\mathrm{d}u}{u^2}-\tfrac{1}{2}\operatorname{arctg}t-\tfrac{t}{2(t^2+1)}+c$$

$$= \tfrac{1}{2}\ln|t-1|+\operatorname{arctg}t-\tfrac{1}{4}\ln|u|+\tfrac{1}{2}\tfrac{u^{-1}}{-1}-\tfrac{t}{2(1+t^2)}+c = \left[\ \tfrac{1}{2}\tfrac{u^{-1}}{-1}=-\tfrac{1}{2u}=-\tfrac{1}{2(1+t^2)}\ \right]$$

$$= \tfrac{1}{4}\ln(t-1)^2+\operatorname{arctg}t-\tfrac{1}{4}\ln(t^2+1)-\tfrac{1}{2(1+t^2)}-\tfrac{t}{2(1+t^2)}+c$$

$$= \tfrac{1}{4}\ln\tfrac{t^2-2t+1}{1+t^2}+\operatorname{arctg}t-\tfrac{1+t}{2(1+t^2)}+c = \tfrac{1}{4}\ln\left(1-\tfrac{2t}{1+t^2}\right)+\operatorname{arctg}t-\tfrac{1+t}{2(1+t^2)}+c$$

$$= \tfrac{1}{4}\ln\left(1-2\sqrt{x-x^2}\right)+\operatorname{arctg}\tfrac{\sqrt{x-x^2}}{x}-\tfrac{x+\sqrt{x-x^2}}{2}+c, \text{ for } x\in(0\,;\,1).$$

---

$$\int\frac{1-x}{x\sqrt{x^2-x}}\,\mathrm{d}x = \left[ \begin{array}{c} \boxed{\text{3rd ES}}\ t=\sqrt{\tfrac{x-1}{x}},\ t^2=\tfrac{x-1}{x},\ t^2x=x-1,\ x=\tfrac{1}{1-t^2},\ \mathrm{d}x=\tfrac{-1\cdot(-2t)}{(1-t^2)^2}\,\mathrm{d}t=\tfrac{2t\,\mathrm{d}t}{|1-t^2|^2} \\ x^2-x=x(x-1)>0,\ x\in(-\infty\,;\,0)\Leftrightarrow t\in(1\,;\,\infty),\ x\in(1\,;\,\infty)\Leftrightarrow t\in(0\,;\,1) \\ x\in R-\langle 0\,;\,1\rangle,\ t=\sqrt{\tfrac{x^2-x}{x^2}}=\tfrac{\sqrt{x^2-x}}{|x|},\ \tfrac{x-1}{x\sqrt{x^2-x}}=\tfrac{x-1}{x\cdot t\,|x|}=\tfrac{t^2}{t\,|x|}=\tfrac{t}{|x|}=t\,|1-t^2| \end{array} \right]$$

$$= -\frac{2\sqrt{x^2-x}}{x}+\ln\left|\frac{\sqrt{x^2-x}+x}{\sqrt{x^2-x}-x}\right|+c, \text{ for } x\in(-\infty\,;\,0)\cup(1\,;\,\infty).$$

$$\int\frac{1-x}{x\sqrt{x^2-x}}\,\mathrm{d}x = \int\frac{t\,|1-t^2|\cdot 2t\,\mathrm{d}t}{|1-t^2|^2} = \int\frac{2t^2\,\mathrm{d}t}{|1-t^2|} = \left[\begin{array}{c} t\in(1\,;\,\infty) \\ x\in(-\infty\,;\,0) \end{array}\right] = 2\int\frac{t^2\,\mathrm{d}t}{t^2-1} = 2\int\frac{t^2-1+1\,\mathrm{d}t}{t^2-1}$$

$$= 2\int\left[1+\tfrac{1}{t^2-1}\right]\mathrm{d}t = 2\left[t+\tfrac{1}{2}\ln\left|\tfrac{t-1}{t+1}\right|\right]+c = 2t+\ln\left|\tfrac{t-1}{t+1}\right|+c = 2\frac{\sqrt{x^2-x}}{-x}+\ln\left|\frac{\frac{\sqrt{x^2-x}}{-x}-1}{\frac{\sqrt{x^2-x}}{-x}+1}\right|+c$$

$$= -\frac{2\sqrt{x^2-x}}{x}+\ln\left|\frac{\sqrt{x^2-x}+x}{\sqrt{x^2-x}-x}\right|+c, \text{ for } x\in(-\infty\,;\,0).$$

$$\int\frac{1-x}{x\sqrt{x^2-x}}\,\mathrm{d}x = \int\frac{t\,|1-t^2|\cdot 2t\,\mathrm{d}t}{|1-t^2|^2} = \int\frac{2t^2\,\mathrm{d}t}{|1-t^2|} = \left[\begin{array}{c} t\in(0\,;\,1) \\ x\in(1\,;\,\infty) \end{array}\right] = 2\int\frac{t^2\,\mathrm{d}t}{1-t^2} = -2\int\frac{t^2-1+1\,\mathrm{d}t}{t^2-1}$$

$$= -2\int\left[1+\tfrac{1}{t^2-1}\right]\mathrm{d}t = -2\left[t+\tfrac{1}{2}\ln\left|\tfrac{t-1}{t+1}\right|\right]+c = -2t-\ln\left|\tfrac{t-1}{t+1}\right|+c = -2t+\ln\left|\tfrac{t+1}{t-1}\right|+c$$

$$= -2\frac{\sqrt{x^2-x}}{x}-\ln\left|\frac{\frac{\sqrt{x^2-x}}{x}+1}{\frac{\sqrt{x^2-x}}{x}-1}\right|+c = -\frac{2\sqrt{x^2-x}}{x}+\ln\left|\frac{\sqrt{x^2-x}+x}{\sqrt{x^2-x}-x}\right|+c, \text{ for } x\in(1\,;\,\infty).$$

---

$$\int\frac{1-x}{x-\sqrt{x^2-x}}\,\mathrm{d}x = \left[\begin{array}{c} x^2-x=x(x-1)>0 \\ x\in(-\infty\,;\,0)\cup(1\,;\,\infty) \end{array}\right] = \int\frac{1-x}{x-\sqrt{x^2-x}}\,\frac{x+\sqrt{x^2-x}}{x+\sqrt{x^2-x}}\,\mathrm{d}x = \left[\ x^2-x=(x-\tfrac{1}{2})-\tfrac{1}{4}\ \right]$$

$$= \int\frac{x-x^2+\sqrt{x^2-x}-x\sqrt{x^2-x}}{x^2-(x^2-x)}\,\mathrm{d}x = \int\left(1-x+\frac{\sqrt{x^2-x}}{x}-\sqrt{(x-\tfrac{1}{2})^2-\tfrac{1}{4}}\right)\mathrm{d}x$$

$$= \left[\begin{array}{l} \text{p. 46}: \int\frac{\sqrt{x^2-x}}{x}\,\mathrm{d}x=\tfrac{1}{2}\ln\left|2x-1-2\sqrt{x^2-x}\right|+\sqrt{x^2-x},x\in(-\infty\,;\,0)\cup\langle 1\,;\,\infty) \\ \text{p. 40}: \int\sqrt{x^2-a^2}\,\mathrm{d}x=\frac{x\sqrt{x^2-a^2}}{2}-\frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right|,\ x\in(-\infty\,;\,-a\rangle\cup\langle a\,;\,\infty),\ a>0 \\ \int\sqrt{(x-\tfrac{1}{2})^2-\tfrac{1}{4}}\,\mathrm{d}x=\frac{(x-\frac{1}{2})\sqrt{(x-\frac{1}{2})^2-\frac{1}{4}}}{2}-\frac{\frac{1}{4}}{2}\ln\left|(x-\tfrac{1}{2})+\sqrt{(x-\tfrac{1}{2})^2-\tfrac{1}{4}}\right| \end{array}\right]$$

$$= x-\tfrac{x^2}{2}+\tfrac{1}{2}\ln\left|2x-1-2\sqrt{x^2-x}\right|+\sqrt{x^2-x}-\left[\frac{(x-\frac{1}{2})\sqrt{x^2-x}}{2}-\tfrac{1}{8}\ln\left|(x-\tfrac{1}{2})+\sqrt{x^2-x}\right|\right]+c_1$$

$$= x-\tfrac{x^2}{2}+\tfrac{1}{2}\ln\left|2x-1-2\sqrt{x^2-x}\right|-(\tfrac{x}{2}-\tfrac{5}{4})\sqrt{x^2-x}+\tfrac{1}{8}\ln\left|\frac{2x-1+2\sqrt{x^2-x}}{2}\right|+c_1$$

$$= \left[\ \frac{2x-1+2\sqrt{x^2-x}}{2}\,\frac{2x-1-2\sqrt{x^2-x}}{2x-1-2\sqrt{x^2-x}}=\frac{1}{2}\,\frac{(4x^2-4x+1)-4(x^2-x)}{2x-1-2\sqrt{x^2-x}}=\frac{1}{2}\,\frac{1}{2x-1-2\sqrt{x^2-x}}\ \right]$$

$$= x-\tfrac{x^2}{2}+\tfrac{1}{2}\ln\left|2x-1-2\sqrt{x^2-x}\right|+\tfrac{5-2x}{4}\sqrt{x^2-x}+\tfrac{1}{8}\ln\left|\frac{1}{2}\,\frac{1}{2x-1-2\sqrt{x^2-x}}\right|+c_1$$

$$= x-\frac{x^2}{2}+\frac{1}{2}\ln\left|2x-1-2\sqrt{x^2-x}\right|+\frac{5-2x}{4}\sqrt{x^2-x}-\frac{1}{8}\ln\left|2x-1-2\sqrt{x^2-x}\right|+\frac{1}{8}\ln\frac{1}{2}+c_1$$
$$=\left[\ \frac{1}{8}\ln\frac{1}{2}+c_1=c\ \right]=x-\frac{x^2}{2}+\frac{3}{8}\ln\left|2x-1-2\sqrt{x^2-x}\right|+\frac{5-2x}{4}\sqrt{x^2-x}+c,$$
for $x\in(-\infty\,;\,0)\cup(1\,;\,\infty)$.

---

$$\int\frac{\mathrm{d}x}{\sqrt{\mathrm{e}^{2x}+\mathrm{e}^x+1}}=\left[\begin{array}{c}\mathrm{e}^x=t,\ x\in R,\ t>0\\ x=\ln t,\ \mathrm{d}x=\frac{\mathrm{d}t}{t}\end{array}\right]=\int\frac{\mathrm{d}t}{t\sqrt{t^2+t+1}}$$
$$=\left[\begin{array}{c}t=\frac{1}{u},\ \sqrt{t^2+t+1}=\sqrt{\frac{1}{u^2}+\frac{1}{u}+1}=\frac{\sqrt{1+u+u^2}}{u}\\ \mathrm{d}t=\frac{-\mathrm{d}u}{u^2},\ \sqrt{1+u+u^2}=\sqrt{1+\frac{1}{t}+\frac{1}{t^2}}=\frac{\sqrt{t^2+t+1}}{t}\end{array}\right]=-\int\frac{\frac{\mathrm{d}u}{u^2}}{\frac{1}{u}\,\frac{\sqrt{1+u+u^2}}{u}}=-\int\frac{\mathrm{d}u}{\sqrt{1+u+u^2}}$$
$$=-\int\frac{\mathrm{d}u}{\sqrt{(u+\frac{1}{2})^2+\frac{3}{4}}}=-\ln\left(u+\frac{1}{2}+\sqrt{(u+\frac{1}{2})^2+\frac{3}{4}}\right)+c_1=-\ln\left(u+\frac{1}{2}+\sqrt{1+u+u^2}\right)+c_1$$
$$=c_1-\ln\left(\frac{1}{t}+\frac{1}{2}+\frac{\sqrt{t^2+t+1}}{t}\right)=c_1-\ln\frac{2+t+2\sqrt{t^2+t+1}}{2t}=c_1-\ln\frac{2+t+2\sqrt{t^2+t+1}}{t}-\ln\frac{1}{2}$$
$$=\left[\ c_1-\ln\frac{1}{2}=c\ \right]=-\ln\frac{2+\mathrm{e}^x+2\sqrt{\mathrm{e}^{2x}+\mathrm{e}^x+1}}{\mathrm{e}^x}+c=x-\ln\left(2+\mathrm{e}^x+2\sqrt{\mathrm{e}^{2x}+\mathrm{e}^x+1}\right)+c,$$
for $x\in R$.

---

$$\int\sqrt{\frac{1-\mathrm{e}^x}{1+\mathrm{e}^x}}\,\mathrm{d}x=\left[\begin{array}{c}\mathrm{e}^x>0,\ 1-\mathrm{e}^x\geq 0,\ 1\geq\mathrm{e}^x,\ x\in(-\infty\,;\,0\rangle\\ \mathrm{e}^x=t,\ x=\ln t,\ \mathrm{d}x=\frac{\mathrm{d}t}{t},\ t\in(0\,;\,1\rangle\end{array}\right]=\int\sqrt{\frac{1-t}{1+t}}\,\frac{\mathrm{d}t}{t}$$
$$=\left[\begin{array}{l}u=\sqrt{\frac{1-t}{1+t}},\ u^2=\frac{1-t}{1+t},\ u^2+u^2t=1-t,\ t=\frac{1-u^2}{1+u^2},\ \mathrm{d}t=\frac{-2u(1+u^2)-2u(1-u^2)}{(1+u^2)^2}\,\mathrm{d}u=-\frac{4u\,\mathrm{d}u}{(1+u^2)^2}\\ 1\pm u=1\pm\frac{\sqrt{1-t}}{\sqrt{1+t}}=\frac{\sqrt{1+t}\pm\sqrt{1-t}}{\sqrt{1+t}}>0,\ \frac{4u^2}{(u^2-1)(u^2+1)}=\frac{2u^2-2+2u^2+2}{(u^2-1)(u^2+1)}=\frac{2}{u^2+1}+\frac{2}{u^2-1},\ u\in\langle 0\,;\,1)\\ \frac{1-u}{1+u}=\frac{\sqrt{1+t}-\sqrt{1-t}}{\sqrt{1+t}+\sqrt{1-t}}=\frac{\sqrt{1+t}-\sqrt{1-t}}{\sqrt{1+t}+\sqrt{1-t}}\cdot\frac{\sqrt{1+t}-\sqrt{1-t}}{\sqrt{1+t}-\sqrt{1-t}}=\frac{1+t-2\sqrt{1-t^2}+1-t}{1+t-(1-t)}=\frac{2-2\sqrt{1-t^2}}{2t}=\frac{1-\sqrt{1-t^2}}{t}\end{array}\right]$$
$$=\int\frac{1+u^2}{1-u^2}\,\frac{-4u\cdot u\,\mathrm{d}u}{(1+u^2)^2}=\int\frac{4u^2\,\mathrm{d}u}{(u^2-1)(u^2+1)}=\int\left[\frac{2}{u^2+1}+\frac{2}{u^2-1}\right]\mathrm{d}x=2\,\mathrm{arctg}\,u+\frac{2}{2}\ln\left|\frac{u-1}{u+1}\right|+c$$
$$=2\,\mathrm{arctg}\sqrt{\frac{1-t}{1+t}}+\ln\frac{1-\sqrt{1-t^2}}{t}+c=2\,\mathrm{arctg}\sqrt{\frac{1-\mathrm{e}^x}{1+\mathrm{e}^x}}+\ln\frac{1-\sqrt{1-\mathrm{e}^{2x}}}{\mathrm{e}^x}+c$$
$$=2\,\mathrm{arctg}\sqrt{\frac{1-\mathrm{e}^x}{1+\mathrm{e}^x}}+\ln\left(1-\sqrt{1-\mathrm{e}^{2x}}\right)+x+c,$$ for $x\in(-\infty\,;\,0\rangle$.

---

$$\int\frac{\mathrm{d}x}{a^2\cos^2x+b^2\sin^2x}=\left[\begin{array}{c}\sin^2x=\frac{1-\cos 2x}{2}\\ \cos^2x=\frac{1+\cos 2x}{2}\end{array}\right]=\int\frac{2\,\mathrm{d}x}{a^2(1+\cos 2x)+b^2(1-\cos 2x)}=\left[\begin{array}{c}2x=t\\ 2\,\mathrm{d}x=\mathrm{d}t\end{array}\right]$$
$$=\int\frac{\mathrm{d}t}{(a^2+b^2)+(a^2-b^2)\cos t}=\left[\begin{array}{c}\boxed{\text{UTS}}\ \ u=\mathrm{tg}\,\frac{t}{2}=\mathrm{tg}\,x\\ \mathrm{d}t=\frac{2\,\mathrm{d}u}{u^2+1},\cos t=\frac{1-u^2}{u^2+1}\end{array}\right]=\int\frac{\frac{2\,\mathrm{d}u}{u^2+1}}{(a^2+b^2)+(a^2-b^2)\frac{1-u^2}{u^2+1}}$$
$$=\int\frac{2\,\mathrm{d}u}{(a^2+b^2)(u^2+1)+(a^2-b^2)(1-u^2)}=\int\frac{2\,\mathrm{d}u}{2b^2u^2+2a^2}=\frac{1}{b^2}\int\frac{\mathrm{d}u}{u^2+\frac{a^2}{b^2}}=\frac{1}{b^2}\,\frac{b}{a}\,\mathrm{arctg}\,\frac{bu}{a}+c$$
$$=\frac{1}{ab}\,\mathrm{arctg}\left(\frac{b}{a}\,\mathrm{tg}\,\frac{t}{2}\right)+c=\frac{1}{ab}\,\mathrm{arctg}\left(\frac{b}{a}\,\mathrm{tg}\,x\right)+c,$$ for $x\in R,\ a>0,\ b>0$.

---

$$\int\frac{\mathrm{d}x}{a^2\cos^2x-b^2\sin^2x}=\left[\begin{array}{c}\sin^2x=\frac{1-\cos 2x}{2}\\ \cos^2x=\frac{1+\cos 2x}{2}\end{array}\right]=\int\frac{2\,\mathrm{d}x}{a^2(1+\cos 2x)-b^2(1-\cos 2x)}=\left[\begin{array}{c}2x=t\\ 2\,\mathrm{d}x=\mathrm{d}t\end{array}\right]$$
$$=\int\frac{\mathrm{d}t}{(a^2-b^2)+(a^2+b^2)\cos t}=\left[\begin{array}{c}\boxed{\text{UTS}}\ \ u=\mathrm{tg}\,\frac{t}{2}=\mathrm{tg}\,x\\ \mathrm{d}t=\frac{2\,\mathrm{d}u}{u^2+1},\cos t=\frac{1-u^2}{u^2+1}\end{array}\right]=\int\frac{\frac{2\,\mathrm{d}u}{u^2+1}}{(a^2-b^2)+(a^2+b^2)\frac{1-u^2}{u^2+1}}$$
$$=\int\frac{2\,\mathrm{d}u}{(a^2-b^2)(u^2+1)+(a^2+b^2)(1-u^2)}=\int\frac{2\,\mathrm{d}u}{-2b^2u^2+2a^2}=-\frac{1}{b^2}\int\frac{\mathrm{d}u}{u^2-\frac{a^2}{b^2}}=-\frac{1}{b^2}\,\frac{b}{2a}\ln\left|\frac{u-\frac{a}{b}}{u+\frac{a}{b}}\right|+c$$
$$=-\frac{1}{2ab}\ln\left|\frac{bu-a}{bu+a}\right|+c=-\frac{1}{2ab}\ln\left|\frac{b\,\mathrm{tg}\,x-a}{b\,\mathrm{tg}\,x+a}\right|+c,$$ for $x\in R,\ a>0,\ b>0$.

---

$$\int x\,\mathrm{tg}^2 x\,\mathrm{d}x = \int \frac{x\sin^2 x}{\cos^2 x}\,\mathrm{d}x = \int \frac{x(1-\cos^2 x)}{\cos^2 x}\,\mathrm{d}x = \int \frac{x\,\mathrm{d}x}{\cos^2 x} - \int x\,\mathrm{d}x = \int \frac{x\,\mathrm{d}x}{\cos^2 x} - \frac{x^2}{2}$$
$$= \left[\begin{array}{l|l} u = x & u' = 1 \\ v' = \frac{1}{\cos^2 x} & v = \mathrm{tg}\,x = \frac{\sin x}{\cos x}\end{array}\right] = x\,\mathrm{tg}\,x - \int \frac{\sin x}{\cos x}\,\mathrm{d}x - \frac{x^2}{2} = x\,\mathrm{tg}\,x + \int \frac{-\sin x}{\cos x}\,\mathrm{d}x - \frac{x^2}{2}$$
$$= x\,\mathrm{tg}\,x + \ln|\cos x| - \frac{x^2}{2} + c, \text{ for } x\in R,\ x\neq \frac{(2k+1)\pi}{2},\ k\in Z.$$

---

$$\int \frac{\mathrm{d}x}{1+\sqrt[3]{x}} = \left[\begin{array}{c} x = t^3,\ t = \sqrt[3]{x},\ x\geq 0 \\ \mathrm{d}x = 3t^2\,\mathrm{d}t,\ t\geq 0\end{array}\right] = \int \frac{3t^2\,\mathrm{d}t}{1+t} = 3\int \frac{t^2\,\mathrm{d}t}{1+t} = 3\int \frac{t^2+t-t-1+1}{1+t}\,\mathrm{d}t$$
$$= 3\int \left[t - 1 + \frac{1}{t+1}\right]\mathrm{d}t = \frac{3t^2}{2} - 3t + 3\ln|t+1| + c = \frac{3t^2}{2} - 3t + 3\ln(t+1) + c$$
$$= \frac{3\sqrt[3]{x^2}}{2} - 3\sqrt[3]{x} + 3\ln(\sqrt[3]{x}+1) + c, \text{ for } x\in\langle 0\,;\,\infty).$$

---

$$\int \frac{\mathrm{d}x}{\sqrt[3]{x}+\sqrt[4]{x}} = \left[\begin{array}{c} x = t^{12},\ t = \sqrt[12]{x},\ \mathrm{d}x = 12t^{11}\,\mathrm{d}t,\ x>0 \\ \sqrt[3]{x} = \sqrt[3]{t^{12}} = t^4,\ \sqrt[4]{x} = \sqrt[4]{t^{12}} = t^3,\ t>0\end{array}\right] = \int \frac{12t^{11}\,\mathrm{d}t}{t^4+t^3} = 12\int \frac{t^8\,\mathrm{d}t}{t+1}$$
$$= 12\int \frac{t^8+t^7-t^7-t^6+t^6+t^5-t^5-t^4+t^4+t^3-t^3-t^2+t^2+t-t-1+1}{t+1}\,\mathrm{d}t$$
$$= 12\int \left[t^7 - t^6 + t^5 - t^4 + t^3 - t^2 + t - 1 + \frac{1}{t+1}\right]\mathrm{d}t$$
$$= 12\left[\frac{t^8}{8} - \frac{t^7}{7} + \frac{t^6}{6} - \frac{t^5}{5} + \frac{t^4}{4} - \frac{t^3}{3} + \frac{t^2}{2} - t + \ln|t+1|\right] + c = \left[\begin{array}{c} t^n = \sqrt[12]{x^n} \\ n = 1,2,\ldots,8\end{array}\right]$$
$$= \frac{3\sqrt[3]{x^2}}{2} - \frac{12\sqrt[12]{x^7}}{7} + 2\sqrt{x} - \frac{12\sqrt[12]{x^5}}{5} + 3\sqrt[3]{x} - 4\sqrt[4]{x} + 6\sqrt[6]{x} - 12\sqrt[12]{x} + 12\ln|\sqrt[12]{x}+1| + c,$$
for $x\in(0\,;\,\infty)$.

---

$$\int \frac{\mathrm{d}x}{\sqrt[n]{1-x^n}} = \left[\begin{array}{c} t = \sqrt[n]{\frac{1}{x^n}-1} = \sqrt[n]{\frac{1-x^n}{x^n}} = \frac{\sqrt[n]{1-x^n}}{x},\ t^n = \frac{1}{x^n}-1,\ x^n = \frac{1}{t^n+1},\ x = \frac{1}{\sqrt[n]{t^n+1}},\ x\in(0\,;\,1) \\ \mathrm{d}x = -\frac{1}{n}(t^n+1)^{-\frac{1}{n}-1}nt^{n-1}\,\mathrm{d}t = \frac{-t^{n-1}\,\mathrm{d}t}{\sqrt[n]{(t^n+1)^{n+1}}},\ \sqrt[n]{1-x^n} = tx = \frac{t}{\sqrt[n]{t^n+1}},\ t\in(0\,;\,\infty)\end{array}\right]$$
$$= \int \frac{\sqrt[n]{t^n+1}}{t}\,\frac{-t^{n-1}\,\mathrm{d}t}{\sqrt[n]{(t^n+1)^{n+1}}} = -\int \frac{t^{n-2}\,\mathrm{d}t}{\sqrt[n]{(t^n+1)^n}} = -\int \frac{t^{n-2}\,\mathrm{d}t}{t^n+1} = \cdots \text{ [Partial fractions]},$$
for $x\in(0\,;\,1)$, $n = 2,3,4,\ldots$.

---

$$\int \frac{\mathrm{d}x}{\sqrt[n]{x^n+1}} = \left[\begin{array}{c} t = \sqrt[n]{1+\frac{1}{x^n}} = \sqrt[n]{\frac{x^n+1}{x^n}} = \frac{\sqrt[n]{x^n+1}}{x},\ t^n = 1+\frac{1}{x^n},\ x^n = \frac{1}{t^n-1},\ x = \frac{1}{\sqrt[n]{t^n-1}},\ x\in(0\,;\,\infty) \\ \mathrm{d}x = -\frac{1}{n}(t^n-1)^{-\frac{1}{n}-1}nt^{n-1}\,\mathrm{d}t = \frac{-t^{n-1}\,\mathrm{d}t}{\sqrt[n]{(t^n-1)^{n+1}}},\ \sqrt[n]{x^n+1} = tx = \frac{t}{\sqrt[n]{t^n-1}},\ t\in(1\,;\,\infty)\end{array}\right]$$
$$= \int \frac{\sqrt[n]{t^n-1}}{t}\,\frac{-t^{n-1}\,\mathrm{d}t}{\sqrt[n]{(t^n-1)^{n+1}}} = -\int \frac{t^{n-2}\,\mathrm{d}t}{\sqrt[n]{(t^n-1)^n}} = -\int \frac{t^{n-2}\,\mathrm{d}t}{t^n-1} = \cdots \text{ [Partial fractions]},$$
for $x\in(0\,;\,\infty)$, $n = 2,3,4,\ldots$.

---

$$\int \frac{\mathrm{d}x}{\sqrt[n]{x^n-1}} = \left[\begin{array}{c} t = \sqrt[n]{1-\frac{1}{x^n}} = \sqrt[n]{\frac{x^n-1}{x^n}} = \frac{\sqrt[n]{x^n-1}}{x},\ t^n = 1-\frac{1}{x^n},\ x^n = \frac{1}{1-t^n},\ x = \frac{1}{\sqrt[n]{1-t^n}},\ x\in(1\,;\,\infty) \\ \mathrm{d}x = -\frac{1}{n}(1-t^n)^{-\frac{1}{n}-1}(-nt^{n-1})\,\mathrm{d}t = \frac{t^{n-1}\,\mathrm{d}t}{\sqrt[n]{(1-t^n)^{n+1}}},\ \sqrt[n]{x^n-1} = tx = \frac{t}{\sqrt[n]{1-t^n}},\ t\in(0\,;\,1)\end{array}\right]$$
$$= \int \frac{\sqrt[n]{1-t^n}}{t}\,\frac{t^{n-1}\,\mathrm{d}t}{\sqrt[n]{(1-t^n)^{n+1}}} = \int \frac{t^{n-2}\,\mathrm{d}t}{\sqrt[n]{(1-t^n)^n}} = \int \frac{t^{n-2}\,\mathrm{d}t}{1-t^n} = \cdots \text{ [Partial fractions]},$$
for $x\in(1\,;\,\infty)$, $n = 2,3,4,\ldots$.

---

$$\int\frac{dx}{\sqrt{x^2+1}}=\ln\left(\sqrt{x^2+1}+x\right)+c, \text{ for } x\in R.$$

$$\int\frac{dx}{\sqrt{x^2+1}}=\left[\begin{array}{c} t=\sqrt{1+\frac{1}{x^2}}=\frac{\sqrt{x^2+1}}{x},\ t^2=1+\frac{1}{x^2},\ x^2=\frac{1}{t^2-1},\ x=\frac{1}{\sqrt{t^2-1}},\ x\in(0\,;\,\infty) \\ dx=-\frac{1}{2}(t^2-1)^{-\frac{1}{2}-1}2t\,dt=\frac{-t\,dt}{\sqrt{(t^2-1)^3}},\ \sqrt{x^2+1}=tx=\frac{t}{\sqrt{t^2-1}},\ t\in(1\,;\,\infty) \\ t\pm1=\frac{\sqrt{x^2+1}}{x}\pm1=\frac{\sqrt{x^2+1}\pm x}{x},\ \frac{t+1}{t-1}=\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}-x}=\frac{(\sqrt{x^2+1}+x)^2}{(x^2+1)-x^2}=(\sqrt{x^2+1}+x)^2 \end{array}\right]$$

$$=\int\frac{\sqrt{t^2-1}}{t}\,\frac{-t\,dt}{\sqrt{(t^2-1)^3}}=-\int\frac{dt}{t^2-1}=-\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|+c=\frac{1}{2}\ln\frac{t+1}{t-1}+c$$

$$=\frac{1}{2}\ln\left(\sqrt{x^2+1}+x\right)^2+c=\ln\left(\sqrt{x^2+1}+x\right)+c, \text{ for } x>0.$$

$$\int\frac{dx}{\sqrt{x^2+1}}=\left[\begin{array}{c} x=-u,\ x<0 \\ dx=-du,\ u>0 \end{array}\right]=-\int\frac{du}{\sqrt{u^2+1}}=-\ln\left(\sqrt{u^2+1}+u\right)+c$$

$$=-\ln\left(\sqrt{x^2+1}-x\right)+c=\left[\ \sqrt{x^2+1}-x=(\sqrt{x^2+1}-x)\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}=\frac{(x^2+1)-x^2}{\sqrt{x^2+1}+x}=\frac{1}{\sqrt{x^2+1}+x}\ \right]$$

$$=-\ln\frac{1}{\sqrt{x^2+1}+x}+c=\ln\left(\sqrt{x^2+1}+x\right)+c, \text{ for } x<0.$$

---

$$\int\frac{dx}{\sqrt[3]{x^3+1}}=-\frac{1}{2}\ln\left(\sqrt[3]{x^3+1}-x\right)-\frac{\sqrt{3}}{3}\operatorname{arctg}\frac{2\sqrt[3]{x^3+1}+x}{x\sqrt{3}}+c, \text{ for } x\in(-1\,;\,\infty)-\{0\}.$$

$$\int\frac{dx}{\sqrt[3]{x^3+1}}=\left[\begin{array}{c} t=\sqrt[3]{1+\frac{1}{x^3}}=\frac{\sqrt[3]{x^3+1}}{x},\ t^3=1+\frac{1}{x^3},\ x^3=\frac{1}{t^3-1},\ x=\frac{1}{\sqrt[3]{t^3-1}},\ x\in(0\,;\,\infty) \\ dx=-\frac{1}{3}(t^3-1)^{-\frac{1}{3}-1}3t^2\,dt=\frac{-t^2\,dt}{\sqrt[3]{(t^3-1)^4}},\ \sqrt[3]{x^3+1}=tx=\frac{t}{\sqrt[3]{t^3-1}},\ t\in(1\,;\,\infty) \end{array}\right]$$

$$=\int\frac{\sqrt[3]{t^3-1}}{t}\,\frac{-t^2\,dt}{\sqrt[3]{(t^3-1)^4}}=-\int\frac{t\,dt}{t^3-1}=\left[\begin{array}{c} \frac{t}{t^3-1}=\frac{t}{(t-1)(t^2+t+1)}=\frac{A}{t-1}+\frac{Bt+C}{t^2+t+1} \\ A=\frac{1}{3},\ B=-\frac{1}{3},\ C=\frac{1}{3} \end{array}\right]$$

$$=-\frac{1}{3}\int\left[\frac{1}{t-1}+\frac{-t+1}{t^2+t+1}\right]dt=-\frac{1}{3}\int\frac{dt}{t-1}+\frac{1}{6}\int\frac{2t-2}{t^2+t+1}\,dt=-\frac{1}{3}\int\frac{dt}{t-1}+\frac{1}{6}\int\frac{2t+1-3}{t^2+t+1}\,dt$$

$$=-\frac{1}{3}\int\frac{dt}{t-1}+\frac{1}{6}\int\frac{2t+1}{t^2+t+1}\,dt-\frac{3}{6}\int\frac{dt}{(t+\frac{1}{2})^2+\frac{3}{4}}=\left[\ t^2+t+1=(t+\tfrac{1}{2})^2+\tfrac{3}{4}>0\ \right]$$

$$=-\frac{1}{3}\ln(t-1)+\frac{1}{6}\ln\left(t^2+t+1\right)-\frac{1}{2}\,\frac{1}{\frac{\sqrt{3}}{2}}\operatorname{arctg}\frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}}+c$$

$$=\left[\ \frac{1}{6}\ln(t^2+t+1)=\frac{1}{6}\ln\frac{(t^2+t+1)(t-1)}{t-1}=\frac{1}{6}\ln\frac{t^3-1}{t-1}=\frac{1}{6}\ln(t^3-1)-\frac{1}{6}\ln(t-1)\ \right]$$

$$=-\frac{2}{6}\ln(t-1)+\frac{1}{6}\ln\left(t^3-1\right)-\frac{1}{6}\ln(t-1)-\frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2t+1}{\sqrt{3}}+c$$

$$=-\frac{1}{2}\ln(t-1)+\frac{1}{6}\ln\left(t^3-1\right)-\frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2t+1}{\sqrt{3}}+c=\left[\begin{array}{c} \frac{1}{6}\ln(t^3-1)=\frac{1}{6}\ln\frac{1}{x^3}=-\frac{3}{6}\ln x \\ 2t+1=2\frac{\sqrt[3]{x^3+1}}{x}+1=\frac{2\sqrt[3]{x^3+1}+x}{x} \end{array}\right]$$

$$=-\frac{1}{2}\ln\frac{\sqrt[3]{x^3+1}-x}{x}-\frac{1}{2}\ln x-\frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2\sqrt[3]{x^3+1}+x}{x\sqrt{3}}+c$$

$$=-\frac{1}{2}\ln\left(\sqrt[3]{x^3+1}-x\right)-\frac{\sqrt{3}}{3}\operatorname{arctg}\frac{2\sqrt[3]{x^3+1}+x}{x\sqrt{3}}+c, \text{ for } x\in(0\,;\,\infty).$$

$$\int\frac{dx}{\sqrt[3]{x^3+1}}=\left[\begin{array}{c} x=-u,\ x^3+1=1-u^3 \\ dx=-du,\ x\in(-1\,;\,0),\ u\in(0\,;\,1) \end{array}\right]=-\int\frac{du}{\sqrt[3]{1-u^3}}$$

$$=\left[\ \text{p. 57}:\int\frac{du}{\sqrt[3]{1-u^3}}=\frac{1}{2}\ln\left(\sqrt[3]{1-u^3}+u\right)-\frac{\sqrt{3}}{3}\operatorname{arctg}\frac{2\sqrt[3]{1-u^3}-u}{u\sqrt{3}},\ u\in(0\,;\,1)\ \right]$$

$$=-\frac{1}{2}\ln\left(\sqrt[3]{1-u^3}+u\right)+\frac{\sqrt{3}}{3}\operatorname{arctg}\frac{2\sqrt[3]{1-u^3}-u}{u\sqrt{3}}+c$$

$$=-\frac{1}{2}\ln\left(\sqrt[3]{x^3+1}-x\right)+\frac{\sqrt{3}}{3}\operatorname{arctg}\frac{2\sqrt[3]{x^3+1}+x}{-x\sqrt{3}}+c$$

$$=-\frac{1}{2}\ln\left(\sqrt[3]{x^3+1}-x\right)-\frac{\sqrt{3}}{3}\operatorname{arctg}\frac{2\sqrt[3]{x^3+1}+x}{x\sqrt{3}}+c, \text{ for } x\in(-1\,;\,0).$$

---

$$\int\frac{\mathrm{d}x}{\sqrt[4]{x^4+1}}=\frac14\ln\frac{\sqrt[4]{x^4+1}+x}{\sqrt[4]{x^4+1}-x}-\frac12\operatorname{arctg}\frac{\sqrt[4]{x^4+1}}{x}+c,\ \text{for } x\in R-\{0\}.$$

$$\int\frac{\mathrm{d}x}{\sqrt[4]{x^4+1}}=\left[\begin{array}{c} t=\sqrt[4]{1+\frac{1}{x^4}}=\frac{\sqrt[4]{x^4+1}}{x},\ t^4=1+\frac{1}{x^4},\ x^4=\frac{1}{t^4-1},\ x=\frac{1}{\sqrt[4]{t^4-1}},\ x\in(0\,;\,\infty)\\ \mathrm{d}x=-\frac14(t^4-1)^{-\frac14-1}4t^3\,\mathrm{d}t=\frac{-t^3\,\mathrm{d}t}{\sqrt[4]{(t^4-1)^5}},\ \sqrt[4]{x^4+1}=tx=\frac{t}{\sqrt[4]{t^4-1}},\ t\in(1\,;\,\infty)\end{array}\right]$$

$$=\int\frac{\sqrt[4]{t^4-1}}{t}\,\frac{-t^3\,\mathrm{d}t}{\sqrt[4]{(t^4-1)^5}}=-\frac12\int\frac{2t^2\,\mathrm{d}t}{t^4-1}=-\frac12\int\frac{t^2+1+t^2-1\,\mathrm{d}t}{(t^2-1)(t^2+1)}=-\frac12\int\left[\frac{1}{t^2-1}+\frac{1}{t^2+1}\right]\mathrm{d}t$$

$$=-\frac12\,\frac12\ln\left|\frac{t-1}{t+1}\right|-\frac12\operatorname{arctg}t+c=\left[\ |t\pm1|=t\pm1=\frac{\sqrt[4]{x^4+1}}{x}\pm1=\frac{\sqrt[4]{x^4+1}\pm x}{x}\ \right]$$

$$=\frac14\ln\frac{t+1}{t-1}-\frac12\operatorname{arctg}t+c=\frac14\ln\frac{\sqrt[4]{x^4+1}+x}{\sqrt[4]{x^4+1}-x}-\frac12\operatorname{arctg}\frac{\sqrt[4]{x^4+1}}{x}+c,\ \text{for } x>0.$$

$$\int\frac{\mathrm{d}x}{\sqrt[4]{x^4+1}}=\left[\begin{array}{c} x=-u,\ x\in(-\infty\,;\,0)\\ \mathrm{d}x=-\,\mathrm{d}u,\ u\in(0\,;\,\infty)\end{array}\right]=-\int\frac{\mathrm{d}u}{\sqrt[4]{u^4+1}}=-\frac14\ln\frac{\sqrt[4]{u^4+1}+u}{\sqrt[4]{u^4+1}-u}+\frac12\operatorname{arctg}\frac{\sqrt[4]{u^4+1}}{u}+c$$

$$=-\frac14\ln\frac{\sqrt[4]{x^4+1}-x}{\sqrt[4]{x^4+1}+x}+\frac12\operatorname{arctg}\frac{\sqrt[4]{x^4+1}}{-x}+c=\frac14\ln\frac{\sqrt[4]{x^4+1}+x}{\sqrt[4]{x^4+1}-x}-\frac12\operatorname{arctg}\frac{\sqrt[4]{x^4+1}}{x}+c,\ \text{for } x<0.$$

---

$$\int\frac{\mathrm{d}x}{\sqrt{x^2-1}}=\ln\left|\sqrt{x^2-1}+x\right|+c,\ \text{for } x\in(-\infty\,;\,-1)\cup(1\,;\,\infty).$$

$$\int\frac{\mathrm{d}x}{\sqrt{x^2-1}}=\left[\begin{array}{c} t=\sqrt{1-\frac{1}{x^2}}=\sqrt{\frac{x^2-1}{x^2}}=\frac{\sqrt{x^2-1}}{x},\ t^2=1-\frac{1}{x^2},\ x^2=\frac{1}{1-t^2},\ x=\frac{1}{\sqrt{1-t^2}},\ x\in(1\,;\,\infty)\\ \mathrm{d}x=-\frac12(1-t^2)^{-\frac12-1}(-2t)\,\mathrm{d}t=\frac{t\,\mathrm{d}t}{\sqrt{(1-t^2)^3}},\ \sqrt{x^2-1}=tx=\frac{t}{\sqrt{1-t^2}},\ t\in(0\,;\,1)\\ t\pm1=\frac{\sqrt{x^2-1}}{x}\pm1=\frac{\sqrt{x^2-1}\pm x}{x},\ \frac{t+1}{t-1}=\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}-x}=\frac{(\sqrt{x^2-1}+x)^2}{(x^2-1)-x^2}=-(\sqrt{x^2-1}+x)^2\end{array}\right]$$

$$=\int\frac{\sqrt{1-t^2}}{t}\,\frac{t\,\mathrm{d}t}{\sqrt{(1-t^2)^3}}=\int\frac{\mathrm{d}t}{1-t^2}=-\int\frac{\mathrm{d}t}{t^2-1}=-\frac12\ln\left|\frac{t-1}{t+1}\right|+c=\frac12\ln\left|\frac{t+1}{t-1}\right|+c$$

$$=\frac12\ln\left|\left(\sqrt{x^2-1}+x\right)^2\right|+c=\frac22\ln\left|\sqrt{x^2-1}+x\right|+c=\ln\left|\sqrt{x^2-1}+x\right|+c,\ \text{for } x>1.$$

$$\int\frac{\mathrm{d}x}{\sqrt{x^2-1}}=\left[\begin{array}{c} x=-u,\ x<-1\\ \mathrm{d}x=-\,\mathrm{d}u,\ u>1\end{array}\right]=-\int\frac{\mathrm{d}u}{\sqrt{u^2-1}}=-\ln\left|\sqrt{u^2-1}+u\right|+c=-\ln\left|\sqrt{x^2-1}-x\right|+c$$

$$=\left[\ \sqrt{x^2-1}-x=(\sqrt{x^2-1}-x)\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}+x}=\frac{(x^2-1)-x^2}{\sqrt{x^2-1}+x}=\frac{-1}{\sqrt{x^2-1}+x}\ \right]=-\ln\left|\frac{1}{\sqrt{x^2-1}+x}\right|+c$$

$$=\ln\left|\sqrt{x^2-1}+x\right|+c,\ \text{for } x<-1.$$

---

$$\int\frac{\mathrm{d}x}{\sqrt[3]{x^3-1}}=\left[\begin{array}{c} t=\sqrt[3]{1-\frac{1}{x^3}}=\sqrt[3]{\frac{x^3-1}{x^3}}=\frac{\sqrt[3]{x^3-1}}{x},\ t^3=1-\frac{1}{x^3},\ x^3=\frac{1}{1-t^3},\ x=\frac{1}{\sqrt[3]{1-t^3}},\ x\in(1\,;\,\infty)\\ \mathrm{d}x=-\frac13(1-t^3)^{-\frac13-1}(-3t^2)\,\mathrm{d}t=\frac{t^2\,\mathrm{d}t}{\sqrt[3]{(1-t^3)^4}},\ \sqrt[3]{x^3-1}=tx=\frac{t}{\sqrt[3]{1-t^3}},\ t\in(0\,;\,1)\end{array}\right]$$

$$=\int\frac{\sqrt[3]{1-t^3}}{t}\,\frac{t^2\,\mathrm{d}t}{\sqrt[3]{(1-t^3)^4}}=\int\frac{t\,\mathrm{d}t}{1-t^3}=-\int\frac{t\,\mathrm{d}t}{t^3-1}=\left[\begin{array}{c}\frac{t}{t^3-1}=\frac{t}{(t-1)(t^2+t+1)}=\frac{A}{t-1}+\frac{Bt+C}{t^2+t+1}\\ A=\frac13,\ B=-\frac13,\ C=\frac13\end{array}\right]$$

$$=-\frac13\int\left[\frac{1}{t-1}+\frac{-t+1}{t^2+t+1}\right]\mathrm{d}t=-\frac13\int\frac{\mathrm{d}t}{t-1}+\frac16\int\frac{2t-2}{t^2+t+1}\,\mathrm{d}t=-\frac13\int\frac{\mathrm{d}t}{t-1}+\frac16\int\frac{2t+1-3}{t^2+t+1}\,\mathrm{d}t$$

$$=-\frac13\int\frac{\mathrm{d}t}{t-1}+\frac16\int\frac{2t+1}{t^2+t+1}\,\mathrm{d}t-\frac36\int\frac{\mathrm{d}t}{(t+\frac12)^2+\frac34}=\left[\ t^2+t+1=(t+\tfrac12)^2+\tfrac34>0\ \right]$$

$$=-\frac13\ln|t-1|+\frac16\ln\left(t^2+t+1\right)-\frac12\,\frac{1}{\frac{\sqrt3}{2}}\operatorname{arctg}\frac{t+\frac12}{\frac{\sqrt3}{2}}+c$$

$$=\left[\ \frac16\ln(t^2+t+1)=\frac16\ln\frac{(t^2+t+1)(t-1)}{t-1}=\frac16\ln\frac{t^3-1}{t-1}=\frac16\ln\frac{1-t^3}{1-t}=\frac16\ln(1-t^3)-\frac16\ln(1-t)\ \right]$$

$$=-\frac26\ln(1-t)+\frac16\ln\left(1-t^3\right)-\frac16\ln(1-t)-\frac{1}{\sqrt3}\operatorname{arctg}\frac{2t+1}{\sqrt3}+c$$

$$= -\tfrac{1}{2}\ln(1-t) + \tfrac{1}{6}\ln\big(1-t^3\big) - \tfrac{1}{\sqrt{3}}\operatorname{arctg}\tfrac{2t+1}{\sqrt{3}} + c = \left[\begin{array}{c} \frac{1}{6}\ln(1-t^3) = \frac{1}{6}\ln\frac{1}{x^3} = -\frac{3}{6}\ln x \\ 2t+1 = 2\frac{\sqrt[3]{x^3-1}}{x}+1 = \frac{2\sqrt[3]{x^3-1}+x}{x} \end{array}\right]$$

$$= -\tfrac{1}{2}\ln\tfrac{x-\sqrt[3]{x^3-1}}{x} - \tfrac{1}{2}\ln x - \tfrac{1}{\sqrt{3}}\operatorname{arctg}\tfrac{2\sqrt[3]{x^3-1}+x}{x\sqrt{3}} + c$$

$$= -\tfrac{1}{2}\ln\big(x-\sqrt[3]{x^3-1}\big) - \tfrac{\sqrt{3}}{3}\operatorname{arctg}\tfrac{2\sqrt[3]{x^3-1}+x}{x\sqrt{3}} + c, \text{ for } x\in(1\,;\,\infty).$$

---

$$\int\frac{dx}{\sqrt[4]{x^4-1}} = \tfrac{1}{4}\ln\tfrac{x+\sqrt[4]{x^4-1}}{x-\sqrt[4]{x^4-1}} - \tfrac{1}{2}\operatorname{arctg}\tfrac{\sqrt[4]{x^4-1}}{x} + c, \text{ for } x\in(-\infty\,;\,-1)\cup(1\,;\,\infty).$$

$$\int\frac{dx}{\sqrt[4]{x^4-1}} = \left[\begin{array}{c} t=\sqrt[4]{1-\frac{1}{x^4}}=\sqrt[4]{\frac{x^4-1}{x^4}}=\frac{\sqrt[4]{x^4-1}}{x},\ t^4=1-\frac{1}{x^4},\ x^4=\frac{1}{1-t^4},\ x=\frac{1}{\sqrt[4]{1-t^4}},\ x\in(1\,;\,\infty) \\ dx=-\frac{1}{4}(1-t^4)^{-\frac{1}{4}-1}(-4t^3)\,dt=\frac{t^3\,dt}{\sqrt[4]{(1-t^4)^5}},\ \sqrt[4]{x^4-1}=tx=\frac{t}{\sqrt[4]{1-t^4}},\ t\in(0\,;\,1) \end{array}\right]$$

$$= \int\frac{\sqrt[4]{1-t^4}}{t}\,\frac{t^3\,dt}{\sqrt[4]{(1-t^4)^5}} = \int\frac{t^2\,dt}{1-t^4} = -\tfrac{1}{2}\int\frac{2t^2\,dt}{t^4-1} = -\tfrac{1}{2}\int\frac{t^2+1+t^2-1\,dt}{(t^2-1)(t^2+1)}$$

$$= -\tfrac{1}{2}\int\Big[\tfrac{1}{t^2-1}+\tfrac{1}{t^2+1}\Big]\,dt = -\tfrac{1}{2}\tfrac{1}{2}\ln\Big|\tfrac{t-1}{t+1}\Big| - \tfrac{1}{2}\operatorname{arctg}t + c$$

$$= \Big[\ |t\pm 1| = 1\pm t = 1\pm\tfrac{\sqrt[4]{x^4-1}}{x} = \tfrac{x\pm\sqrt[4]{x^4-1}}{x}\ \Big] = \tfrac{1}{4}\ln\tfrac{1+t}{1-t} - \tfrac{1}{2}\operatorname{arctg}t + c$$

$$= \tfrac{1}{4}\ln\tfrac{x+\sqrt[4]{x^4-1}}{x-\sqrt[4]{x^4-1}} - \tfrac{1}{2}\operatorname{arctg}\tfrac{\sqrt[4]{x^4-1}}{x} + c, \text{ for } x>1.$$

$$\int\frac{dx}{\sqrt[4]{x^4-1}} = \left[\begin{array}{c} x=-u,\ x\in(-\infty\,;\,-1) \\ dx=-\,du,\ u\in(1\,;\,\infty) \end{array}\right] = -\int\frac{du}{\sqrt[4]{u^4-1}} = -\tfrac{1}{4}\ln\tfrac{u+\sqrt[4]{u^4-1}}{u-\sqrt[4]{u^4-1}} + \tfrac{1}{2}\operatorname{arctg}\tfrac{\sqrt[4]{u^4-1}}{u} + c$$

$$= -\tfrac{1}{4}\ln\tfrac{-x+\sqrt[4]{x^4-1}}{-x-\sqrt[4]{x^4-1}} + \tfrac{1}{2}\operatorname{arctg}\tfrac{\sqrt[4]{x^4-1}}{-x} + c = -\tfrac{1}{4}\ln\tfrac{x-\sqrt[4]{x^4-1}}{x+\sqrt[4]{x^4-1}} - \tfrac{1}{2}\operatorname{arctg}\tfrac{\sqrt[4]{x^4-1}}{x} + c$$

$$= \tfrac{1}{4}\ln\tfrac{x+\sqrt[4]{x^4-1}}{x-\sqrt[4]{x^4-1}} - \tfrac{1}{2}\operatorname{arctg}\tfrac{\sqrt[4]{x^4-1}}{x} + c, \text{ for } x<-1.$$

---

$$\int\frac{dx}{\sqrt{1-x^2}} = -\operatorname{arctg}\tfrac{\sqrt{1-x^2}}{x} + c_1, \text{ for } x\in(-1\,;\,0)\cup(0\,;\,1).$$

$$\int\frac{dx}{\sqrt{1-x^2}} = \left[\begin{array}{c} t=\sqrt{\frac{1}{x^2}-1}=\sqrt{\frac{1-x^2}{x^2}}=\frac{\sqrt{1-x^2}}{x},\ t^2=\frac{1}{x^2}-1,\ x^2=\frac{1}{t^2+1},\ x=\frac{1}{\sqrt{t^2+1}},\ x\in(0\,;\,1) \\ dx=-\frac{1}{2}(t^2+1)^{-\frac{1}{2}-1}2t\,dt=\frac{-t\,dt}{\sqrt{(t^2+1)^3}},\ \sqrt{1-x^2}=tx=\frac{t}{\sqrt{t^2+1}},\ t\in(0\,;\,\infty) \end{array}\right]$$

$$= \int\frac{\sqrt{t^2+1}}{t}\,\frac{-t\,dt}{\sqrt{(t^2+1)^3}} = -\int\frac{dt}{\sqrt{(t^2+1)^2}} = -\int\frac{dt}{t^2+1} = -\operatorname{arctg}t + c = -\operatorname{arctg}\tfrac{\sqrt{1-x^2}}{x} + c,$$

for $x\in(0\,;\,1)$.

$$\int\frac{dx}{\sqrt{1-x^2}} = \left[\begin{array}{c} x=-u,\ x<-1 \\ dx=-\,du,\ u>1 \end{array}\right] = -\int\frac{du}{\sqrt{1-u^2}} = \operatorname{arctg}\tfrac{\sqrt{1-u^2}}{u} + c = \operatorname{arctg}\tfrac{\sqrt{1-x^2}}{-x} + c$$

$$= -\operatorname{arctg}\tfrac{\sqrt{1-x^2}}{x} + c, \text{ for } x\in(-1\,;\,0).$$

---

$$\int\frac{dx}{\sqrt{1-x^2}} = \left[\begin{array}{c} x=\sin t,\ t=\arcsin x,\ x\in(-1\,;\,1),\ t\in\big(-\frac{\pi}{2}\,;\,\frac{\pi}{2}\big) \\ dx=\cos t\,dt,\ \sqrt{1-x^2}=\sqrt{1-\sin^2 t}=\sqrt{\cos^2 t}=|\cos t|=\cos t \end{array}\right] = \int\frac{\cos t\,dt}{\cos t} = \int dt$$

$$= t + c = \arcsin x + c_3, \text{ for } x\in(-1\,;\,1).$$

---

$$\int\frac{dx}{\sqrt{1-x^2}} = \left[\begin{array}{c} x=\cos t,\ t=\arccos x,\ x\in(-1\,;\,1),\ t\in(0\,;\,\pi) \\ dx=-\sin t\,dt,\ \sqrt{1-x^2}=\sqrt{1-\cos^2 t}=\sqrt{\sin^2 t}=|\sin t|=\sin t \end{array}\right] = -\int\frac{\sin t\,dt}{\sin t}$$

$$= -\int dt = -t + c = -\arccos x + c, \text{ for } x\in(-1\,;\,1).$$

---

$$\int\frac{\mathrm{d}x}{\sqrt{1-x^2}}=\left[\begin{array}{c}\sqrt{1-x^2}=xt+1,\ 1-x^2=x^2t^2+2tx+1,\ x=\frac{-2t}{t^2+1},\ t=\frac{\sqrt{1-x^2}-1}{x},\ x\in(-1\,;\,1)-\{0\}\\ \boxed{\text{2nd ES}}\ \sqrt{1-x^2}=\frac{-2t^2}{t^2+1}+1=\frac{1-t^2}{t^2+1},\mathrm{d}x=\frac{-2(t^2+1)+2t2t}{(t^2+1)^2}\,\mathrm{d}t=\frac{2(t^2-1)\,\mathrm{d}t}{(t^2+1)^2},t\in(-1\,;\,1)-\{0\}\end{array}\right]$$

$$=\int\frac{t^2+1}{1-t^2}\,\frac{2(t^2-1)}{(t^2+1)^2}\,\mathrm{d}t=-2\int\frac{\mathrm{d}t}{t^2+1}=-2\,\mathrm{arctg}\,t+c=-2\,\mathrm{arctg}\,\frac{\sqrt{1-x^2}-1}{x}+c,$$

for $x\in(-1\,;\,1)-\{0\}$.

---

$$\int\frac{\mathrm{d}x}{\sqrt[3]{1-x^3}}=\frac{1}{2}\ln\left(x+\sqrt[3]{1-x^3}\right)-\frac{\sqrt{3}}{3}\,\mathrm{arctg}\,\frac{2\sqrt[3]{1-x^3}-x}{x\sqrt{3}}+c,\ \text{for } x\in(-\infty\,;\,0)\cup(0\,;\,1).$$

$$\int\frac{\mathrm{d}x}{\sqrt[3]{1-x^3}}=\left[\begin{array}{c}t=\sqrt[3]{\frac{1}{x^3}-1}=\sqrt[3]{\frac{1-x^3}{x^3}}=\frac{\sqrt[3]{1-x^3}}{x},\ t^3=\frac{1}{x^3}-1,\ x^3=\frac{1}{t^3+1},\ x=\frac{1}{\sqrt[3]{t^3+1}},\ x\in(0\,;\,1)\\ \mathrm{d}x=-\frac{1}{3}(t^3+1)^{-\frac{1}{3}-1}3t^2\,\mathrm{d}t=\frac{-t^2\,\mathrm{d}t}{\sqrt[3]{(t^3+1)^4}},\ \sqrt[3]{1-x^3}=tx=\frac{t}{\sqrt[3]{t^3+1}},\ t\in(0\,;\,\infty)\end{array}\right]$$

$$=\int\frac{\sqrt[3]{t^3+1}}{t}\,\frac{-t^2\,\mathrm{d}t}{\sqrt[3]{(t^3+1)^4}}=\int\frac{-t\,\mathrm{d}t}{\sqrt[3]{(t^3+1)^3}}=\int\frac{-t\,\mathrm{d}t}{t^3+1}=\left[\begin{array}{c}\frac{-t}{t^3+1}=\frac{-t}{(t+1)(t^2-t+1)}=\frac{A}{t+1}+\frac{Bt+C}{t^2-t+1}\\ A=\frac{1}{3},\ B=-\frac{1}{3},\ C=-\frac{1}{3}\end{array}\right]$$

$$=\frac{1}{3}\int\left[\frac{1}{t+1}-\frac{t+1}{t^2-t+1}\right]\mathrm{d}t=\frac{1}{3}\int\frac{\mathrm{d}t}{t+1}-\frac{1}{6}\int\frac{2t+2}{t^2-t+1}\,\mathrm{d}t=\frac{1}{3}\int\frac{\mathrm{d}t}{t+1}-\frac{1}{6}\int\frac{2t-1+3}{t^2-t+1}\,\mathrm{d}t$$

$$=\frac{1}{3}\int\frac{\mathrm{d}t}{t+1}-\frac{1}{6}\int\frac{2t-1}{t^2-t+1}\,\mathrm{d}t-\frac{3}{6}\int\frac{\mathrm{d}t}{(t-\frac{1}{2})^2+\frac{3}{4}}=\left[\ t^2-t+1=(t-\tfrac{1}{2})^2+\tfrac{3}{4}>0\ \right]$$

$$=\frac{1}{3}\ln|t+1|-\frac{1}{6}\ln\left(t^2-t+1\right)-\frac{1}{2}\,\frac{1}{\frac{\sqrt{3}}{2}}\,\mathrm{arctg}\,\frac{t-\frac{1}{2}}{\frac{\sqrt{3}}{2}}+c$$

$$=\left[\ \frac{1}{6}\ln(t^2-t+1)=\frac{1}{6}\ln\frac{(t^2-t+1)(t+1)}{t+1}=\frac{1}{6}\ln\frac{t^3+1}{t+1}=\frac{1}{6}\ln\frac{t^3+1}{t+1}=\frac{1}{6}\ln(t^3+1)-\frac{1}{6}\ln(t+1)\ \right]$$

$$=\frac{2}{6}\ln(t+1)-\frac{1}{6}\ln\left(t^3+1\right)+\frac{1}{6}\ln(t+1)-\frac{1}{\sqrt{3}}\,\mathrm{arctg}\,\frac{2t-1}{\sqrt{3}}+c$$

$$=\frac{1}{2}\ln(t+1)-\frac{1}{6}\ln\left(t^3+1\right)-\frac{1}{\sqrt{3}}\,\mathrm{arctg}\,\frac{2t-1}{\sqrt{3}}+c=\left[\begin{array}{c}\frac{1}{6}\ln(t^3+1)=\frac{1}{6}\ln\frac{1}{x^3}=-\frac{3}{6}\ln x\\ 2t+1=2\frac{\sqrt[3]{1-x^3}}{x}+1=\frac{2\sqrt[3]{1-x^3}+x}{x}\end{array}\right]$$

$$=\frac{1}{2}\ln\frac{\sqrt[3]{1-x^3}+x}{x}-\frac{1}{2}\ln x-\frac{1}{\sqrt{3}}\,\mathrm{arctg}\,\frac{2\sqrt[3]{1-x^3}-x}{x\sqrt{3}}+c$$

$$=\frac{1}{2}\ln\left(\sqrt[3]{1-x^3}+x\right)-\frac{\sqrt{3}}{3}\,\mathrm{arctg}\,\frac{2\sqrt[3]{1-x^3}-x}{x\sqrt{3}}+c,\ \text{for } x\in(0\,;\,1).$$

$$\int\frac{\mathrm{d}x}{\sqrt[3]{1-x^3}}=\left[\begin{array}{c}x=-u,\ 1-x^3=u^3+1\\ \mathrm{d}x=-\mathrm{d}u,\ x\in(-\infty\,;\,0),\ u\in(0\,;\,\infty)\end{array}\right]=-\int\frac{\mathrm{d}u}{\sqrt[3]{u^3+1}}$$

$$=\left[\ \text{p. 54}:\int\frac{\mathrm{d}u}{\sqrt[3]{u^3+1}}=-\frac{1}{2}\ln\left(\sqrt[3]{u^3+1}-u\right)-\frac{\sqrt{3}}{3}\,\mathrm{arctg}\,\frac{2\sqrt[3]{u^3+1}+u}{u\sqrt{3}},\ u\in(0\,;\,\infty)\ \right]$$

$$=\frac{1}{2}\ln\left(\sqrt[3]{u^3+1}-u\right)+\frac{\sqrt{3}}{3}\,\mathrm{arctg}\,\frac{2\sqrt[3]{u^3+1}+u}{u\sqrt{3}}+c$$

$$=\frac{1}{2}\ln\left(\sqrt[3]{1-x^3}+x\right)+\frac{\sqrt{3}}{3}\,\mathrm{arctg}\,\frac{2\sqrt[3]{1-x^3}-x}{-x\sqrt{3}}+c$$

$$=\frac{1}{2}\ln\left(\sqrt[3]{1-x^3}+x\right)-\frac{\sqrt{3}}{3}\,\mathrm{arctg}\,\frac{2\sqrt[3]{1-x^3}-x}{x\sqrt{3}}+c,\ \text{for } x\in(-\infty\,;\,0).$$

---

$$\int\frac{\mathrm{d}x}{\sqrt[4]{1-x^4}}=\frac{\sqrt{2}}{8}\ln\frac{\sqrt{1-x^4}+x\sqrt[4]{4(1-x^4)}+x^2}{\sqrt{1-x^4}-x\sqrt[4]{4(1-x^4)}+x^2}-\frac{\sqrt{2}}{4}\,\mathrm{arctg}\,\frac{\sqrt[4]{4(1-x^4)}+x}{x}-\frac{\sqrt{2}}{4}\,\mathrm{arctg}\,\frac{\sqrt[4]{4(1-x^4)}-x}{x}+c,$$

for $x\in(-1\,;\,0)\cup(0\,;\,1)$.

$$\int\frac{\mathrm{d}x}{\sqrt[4]{1-x^4}}=\left[\begin{array}{c}t=\sqrt[4]{\frac{1}{x^4}-1}=\sqrt[4]{\frac{1-x^4}{x^4}}=\frac{\sqrt[4]{1-x^4}}{x},\ t^4=\frac{1}{x^4}-1,\ x^4=\frac{1}{t^4+1},\ x=\frac{1}{\sqrt[4]{t^4+1}},\ x\in(0\,;\,1)\\ \mathrm{d}x=-\frac{1}{4}(t^4+1)^{-\frac{1}{4}-1}4t^3\,\mathrm{d}t=\frac{-t^3\,\mathrm{d}t}{\sqrt[4]{(t^4+1)^5}},\ \sqrt[4]{1-x^4}=tx=\frac{t}{\sqrt[4]{t^4+1}},\ t\in(0\,;\,\infty)\end{array}\right]$$

$$=\int\frac{\sqrt[4]{t^4+1}}{t}\,\frac{-t^3\,\mathrm{d}t}{\sqrt[4]{(t^4+1)^5}}=-\int\frac{t^2\,\mathrm{d}t}{\sqrt[4]{(t^4+1)^4}}=-\int\frac{t^2\,\mathrm{d}t}{t^4+1}=-\int\frac{t^2\,\mathrm{d}t}{(t^2+\sqrt{2}t+1)(t^2-\sqrt{2}t+1)}$$

$$= \left[ \ \frac{-t^2}{t^4+1} = \frac{At+B}{t^2+\sqrt{2}t+1} + \frac{Ct+D}{t^2-\sqrt{2}t+1} \ \middle| \ A = \frac{\sqrt{2}}{4},\ B=0,\ C=-\frac{\sqrt{2}}{4},\ D=0 \ \right]$$
$$= \frac{\sqrt{2}}{4}\frac{1}{2}\int \left[ \frac{2t}{t^2+\sqrt{2}t+1} - \frac{2t}{t^2-\sqrt{2}t+1} \right] \mathrm{d}t = \frac{\sqrt{2}}{8}\int \left[ \frac{2t+\sqrt{2}-\sqrt{2}}{t^2+\sqrt{2}t+1} - \frac{2t-\sqrt{2}+\sqrt{2}}{t^2-\sqrt{2}t+1} \right] \mathrm{d}t$$
$$= \frac{\sqrt{2}}{8}\int \frac{2t+\sqrt{2}}{t^2+\sqrt{2}t+1}\,\mathrm{d}t - \frac{\sqrt{2}\sqrt{2}}{8}\int \frac{\mathrm{d}t}{t^2+\sqrt{2}t+1} - \frac{\sqrt{2}}{8}\int \frac{2t-\sqrt{2}}{t^2-\sqrt{2}t+1}\,\mathrm{d}t - \frac{\sqrt{2}\sqrt{2}}{8}\int \frac{\mathrm{d}t}{t^2-\sqrt{2}t+1}$$
$$= \frac{\sqrt{2}}{8}\int \frac{2t+\sqrt{2}}{t^2+\sqrt{2}t+1}\,\mathrm{d}t - \frac{\sqrt{2}}{8}\int \frac{2t-\sqrt{2}}{t^2-\sqrt{2}t+1}\,\mathrm{d}t - \frac{1}{4}\int \frac{\mathrm{d}t}{(t+\frac{\sqrt{2}}{2})^2+\frac{1}{2}} - \frac{1}{4}\int \frac{\mathrm{d}t}{(t-\frac{\sqrt{2}}{2})^2+\frac{1}{2}}$$
$$= \frac{\sqrt{2}}{8}\ln|t^2+\sqrt{2}t+1| - \frac{\sqrt{2}}{8}\ln|t^2-\sqrt{2}t+1| - \frac{1}{4}\frac{1}{\frac{1}{\sqrt{2}}}\operatorname{arctg}\frac{t+\frac{\sqrt{2}}{2}}{\frac{1}{\sqrt{2}}} - \frac{1}{4}\frac{1}{\frac{1}{\sqrt{2}}}\operatorname{arctg}\frac{t-\frac{\sqrt{2}}{2}}{\frac{1}{\sqrt{2}}} + c$$
$$= \frac{\sqrt{2}}{8}\ln(t^2+\sqrt{2}t+1) - \frac{\sqrt{2}}{8}\ln(t^2-\sqrt{2}t+1) - \frac{1}{4}\frac{1}{\frac{1}{\sqrt{2}}}\operatorname{arctg}\frac{t+\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} - \frac{1}{4}\frac{1}{\frac{1}{\sqrt{2}}}\operatorname{arctg}\frac{t-\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} + c$$
$$= \frac{\sqrt{2}}{8}\ln\frac{t^2+\sqrt{2}t+1}{t^2-\sqrt{2}t+1} - \frac{\sqrt{2}}{4}\operatorname{arctg}(\sqrt{2}t+1) - \frac{\sqrt{2}}{4}\operatorname{arctg}(\sqrt{2}t-1) + c$$
$$= \left[ \begin{array}{c} \sqrt{2}t \pm 1 = \sqrt{2}\frac{\sqrt[4]{1-x^4}}{x} \pm 1 = \frac{\sqrt{2}\sqrt[4]{1-x^4}\pm x}{x} = \frac{\sqrt[4]{4(1-x^4)}\pm x}{x} \\ t^2 \pm \sqrt{2}t+1 = \frac{\sqrt{1-x^4}}{x^2} \pm \sqrt{2}\frac{\sqrt[4]{1-x^4}}{x} + 1 = \frac{\sqrt{1-x^4}\pm x\sqrt[4]{4(1-x^4)}+x^2}{x^2} \end{array} \right]$$
$$= \frac{\sqrt{2}}{8}\ln\frac{\sqrt{1-x^4}+x\sqrt[4]{4(1-x^4)}+x^2}{\sqrt{1-x^4}-x\sqrt[4]{4(1-x^4)}+x^2} - \frac{\sqrt{2}}{4}\operatorname{arctg}\frac{\sqrt[4]{4(1-x^4)}+x}{x} - \frac{\sqrt{2}}{4}\operatorname{arctg}\frac{\sqrt[4]{4(1-x^4)}-x}{x} + c,$$

for $x \in (0\,;\,1)$.

$$\int \frac{\mathrm{d}x}{\sqrt[4]{1-x^4}} = \left[ \begin{array}{l} x=-u,\ x<-1 \\ \mathrm{d}x=-\mathrm{d}u,\ u>1 \end{array} \right] = -\int \frac{\mathrm{d}u}{\sqrt{1-u^4}}$$
$$= -\frac{\sqrt{2}}{8}\ln\frac{\sqrt{1-u^4}+u\sqrt[4]{4(1-u^4)}+u^2}{\sqrt{1-u^4}-u\sqrt[4]{4(1-u^4)}+u^2} + \frac{\sqrt{2}}{4}\operatorname{arctg}\frac{\sqrt[4]{4(1-u^4)}+u}{u} + \frac{\sqrt{2}}{4}\operatorname{arctg}\frac{\sqrt[4]{4(1-u^4)}-u}{u} + c$$
$$= -\frac{\sqrt{2}}{8}\ln\frac{\sqrt{1-x^4}-x\sqrt[4]{4(1-x^4)}+x^2}{\sqrt{1-x^4}+x\sqrt[4]{4(1-x^4)}+x^2} + \frac{\sqrt{2}}{4}\operatorname{arctg}\frac{\sqrt[4]{4(1-x^4)}-x}{-x} + \frac{\sqrt{2}}{4}\operatorname{arctg}\frac{\sqrt[4]{4(1-x^4)}+x}{-x} + c$$
$$= \frac{\sqrt{2}}{8}\ln\frac{\sqrt{1-x^4}+x\sqrt[4]{4(1-x^4)}+x^2}{\sqrt{1-x^4}-x\sqrt[4]{4(1-x^4)}+x^2} - \frac{\sqrt{2}}{4}\operatorname{arctg}\frac{\sqrt[4]{4(1-x^4)}-x}{x} - \frac{\sqrt{2}}{4}\operatorname{arctg}\frac{\sqrt[4]{4(1-x^4)}+x}{x} + c,$$

for $x \in (-1\,;\,0)$.

---

$$\int \ln\left(\sqrt{1+x}+\sqrt{1-x}\right)\mathrm{d}x = \left[ \begin{array}{l} v'=1 \\ u=\ln\left(\sqrt{1+x}+\sqrt{1-x}\right) \end{array} \middle| \begin{array}{l} v=x \\ u'=\frac{1}{2x}-\frac{1}{2x\sqrt{1-x^2}} \end{array} \right]$$
$$= \left[ \begin{array}{l} u' = \frac{1}{\sqrt{1+x}+\sqrt{1-x}}\left(\frac{1}{2}\frac{1}{\sqrt{1+x}} - \frac{1}{2}\frac{1}{\sqrt{1-x}}\right) = \frac{1}{\sqrt{1-x}+\sqrt{1+x}}\frac{\sqrt{1-x}-\sqrt{1+x}}{2\sqrt{1-x^2}}\cdot\frac{\sqrt{1-x}-\sqrt{1+x}}{\sqrt{1-x}-\sqrt{1+x}} \\ \quad = \frac{1}{1-x-(1+x)}\frac{1-x-2\sqrt{1-x^2}+1+x}{2\sqrt{1-x^2}} = \frac{2-2\sqrt{1-x^2}}{-2x\cdot2\sqrt{1-x^2}} = \frac{\sqrt{1-x^2}-1}{2x\sqrt{1-x^2}} = \frac{1}{2x} - \frac{1}{2x\sqrt{1-x^2}} \end{array} \right]$$
$$= x\ln\left(\sqrt{1+x}+\sqrt{1-x}\right) - \int\frac{\mathrm{d}x}{2} + \int\frac{\mathrm{d}x}{2\sqrt{1-x^2}}$$
$$= x\ln\left(\sqrt{1+x}+\sqrt{1-x}\right) - \frac{x}{2} + \frac{\arcsin x}{2} + c, \text{ for } x\in\langle -1\,;\,1\rangle - \{0\}.$$

---

$$\int \sin nx\,\mathrm{d}x = \left[ \begin{array}{c} nx=t \\ n\,\mathrm{d}x=\mathrm{d}t \end{array} \right] = \frac{1}{n}\int \sin t\,\mathrm{d}t = -\frac{\cos t}{n}+c = -\frac{\cos nx}{n}+c, \text{ for } x\in R,\ n\in R-\{0\}.$$

---

$$\int x\sin nx\,\mathrm{d}x = \left[ \begin{array}{l} u=x \\ v'=\sin nx \end{array} \middle| \begin{array}{l} u'=1 \\ v=-\frac{\cos nx}{n} \end{array} \right] = -\frac{x\cos nx}{n} + \int\frac{\cos nx}{n}\,\mathrm{d}x$$
$$= -\frac{x\cos nx}{n} + \frac{1}{n}\frac{\sin nx}{n} + c = \frac{\sin nx}{n^2} - \frac{x\cos nx}{n} + c, \text{ for } x\in R,\ n\in R-\{0\}.$$

---

$$\int x \sin nx \, dx = (A+Bx)\sin nx + (C+Dx)\cos nx + c = \frac{\sin nx}{n^2} - \frac{x\cos nx}{n} + c,$$
for $x \in R$, $n \in R-\{0\}$.

**Derivative**

$$0\cos nx + x\sin nx = \left[\int x\sin nx\,dx\right]' = \left[(A+Bx)\sin nx + (C+Dx)\cos nx\right]'$$
$$= B\sin nx + (A+Bx)n\cos nx + D\cos nx - (C+Dx)n\sin nx$$
$$= (B-Cn-Dnx)\sin nx + (An+D+Bnx)\cos nx$$

**Equations**

$\sin nx:\quad 0+x = B-Cn-Dnx,\quad 0=B-Cn,\ 1=-Dn,\quad A=\frac{1}{n^2},\ B=0$

$\cos nx:\quad 0+0x = An+D+Bnx,\quad 0=D+An,\ 0=Bn,\quad C=0,\ D=-\frac{1}{n}$

---

$$\int x^2 \sin nx\,dx = \left[\begin{array}{l|l} u = x^2 & u' = 2x \\ v' = \sin nx & v = -\frac{\cos nx}{n}\end{array}\right] = -\frac{x^2\cos nx}{n} + \frac{2}{n}\int x\cos nx\,dx$$
$$= \left[\begin{array}{l|l} u = x & u' = 1 \\ v' = \cos nx & v = \frac{\sin nx}{n}\end{array}\right] = -\frac{x^2\cos nx}{n} + \frac{2}{n}\left[\frac{x\sin nx}{n} - \frac{1}{n}\int \sin nx\,dx\right]$$
$$= -\frac{x^2\cos nx}{n} + \frac{2}{n}\left[\frac{x\sin nx}{n} - \frac{1}{n}\,\frac{-\cos nx}{n}\right] = \frac{2x\sin nx}{n^2} + \frac{2\cos nx}{n^3} - \frac{x^2\cos nx}{n} + c,$$
for $x \in R$, $n \in R-\{0\}$.

---

$$\int x^2 \sin nx\,dx = (A+Bx+Cx^2)\sin nx + (D+Ex+Fx^2)\cos nx + c$$
$$= \frac{2x}{n^2}\sin nx + \left(\frac{2}{n^3} - \frac{x^2}{n}\right)\cos nx + c,$$ for $x \in R$, $n \in R-\{0\}$.

**Derivative**

$$x^2\sin nx = \left[\int x^2\sin nx\,dx\right]' = \left[(A+Bx+Cx^2)\sin nx + (D+Ex+Fx^2)\cos nx\right]'$$
$$= (B+2Cx)\sin nx + (A+Bx+Cx^2)n\cos nx + (E+2Fx)\cos nx - (D+Ex+Fx^2)n\sin nx$$
$$= [B-Dn+(2C-En)x-Fnx^2]\sin nx + [An+E+(Bn+2F)x+Cnx^2]\cos nx$$

**Equations**

$\sin nx:\quad 0=B-Dn,\ 0=2C-En,\ 1=-Fn,\quad A=0,\ B=\frac{2}{n^2},\ C=0$

$\cos nx:\quad 0=An+E,\ 0=Bn+2F,\ 0=Cn,\quad D=\frac{2}{n^3},\ E=0,\ F=-\frac{1}{n}$

---

$$\int x^3 \sin nx\,dx = (A+Bx+Cx^2+Dx^3)\sin nx + (E+Fx+Gx^2+Hx^3)\cos nx + c$$
$$= \left(-\frac{6}{n^4} + \frac{3x^2}{n^2}\right)\sin nx + \left(\frac{6x}{n^3} - \frac{x^3}{n}\right)\cos nx + c,$$ for $x \in R$, $n \in R-\{0\}$.

**Derivative**

$$x^3\sin nx = \left[\int x^3\sin nx\,dx\right]' = \left[(A+Bx+Cx^2+Dx^3)\sin nx + (E+Fx+Gx^2+Hx^3)\cos nx\right]'$$
$$= (B+2Cx+3Dx^2)\sin nx + (A+Bx+Cx^2+Dx^3)n\cos nx$$
$$+ (F+2Gx+3Hx^2)\cos nx - (E+Fx+Gx^2+Hx^3)n\sin nx$$
$$= [B-En+(2C-Fn)x-(3D-Gn)x^2-Hnx^3]\sin nx$$
$$+ [An+F+(Bn+2G)x+(Cn+3H)x^2+Dnx^3]\cos nx$$

**Equations**

$\sin nx:\ 0=B-En,\ 0=2C-Fn,\ 0=3D-Gn,\ 1=-Hn,\quad A=-\frac{6}{n^4},\ B=D=0,\ C=\frac{3}{n^2}$

$\cos nx:\ 0=An+F,\ 0=Bn+2G,\ 0=Cn+3H,\ 0=Dn,\quad E=G=0,\ F=\frac{6}{n^3},\ H=-\frac{1}{n}$

---

$$\int \cos nx\,dx = \left[\begin{array}{c} nx = t \\ n\,dx = dt\end{array}\right] = \frac{1}{n}\int \cos t\,dt = \frac{\sin t}{n} + c = \frac{\sin nx}{n} + c,$$ for $x \in R$, $n \in R-\{0\}$.

---

$$\int x\cos nx\,dx=\left[\begin{array}{l|l} u=x & u'=1\\ v'=\cos nx & v=\frac{\sin nx}{n}\end{array}\right]=\frac{x\sin nx}{n}-\int\frac{\sin nx}{n}\,dx=\frac{x\sin nx}{n}-\frac{1}{n}\,\frac{-\cos nx}{n}+c$$
$$=\frac{\cos nx}{n^2}+\frac{x\sin nx}{n}+c,\ \text{for } x\in R,\ n\in R-\{0\}.$$

---

$$\int x\cos nx\,dx=(A+Bx)\cos nx+(C+Dx)\sin nx+c=\frac{\cos nx}{n^2}+\frac{x\sin nx}{n}+c,$$
for $x\in R,\ n\in R-\{0\}$.

Derivative

$$0\sin nx+x\cos nx=\left[\int x\cos nx\,dx\right]'=\left[(A+Bx)\cos nx+(C+Dx)\sin nx\right]'$$
$$=B\cos nx-(A+Bx)n\sin nx+D\sin nx+(C+Dx)n\cos nx$$
$$=(B+Cn+Dnx)\cos nx+(-An+D-Bnx)\sin nx$$

Equations

$\cos nx:\quad 0+x=B+Cn+Dnx,\quad 0=B+Cn,\ 1=Dn,\quad A=\frac{1}{n^2},\ B=0$

$\sin nx:\quad 0+0x=-An+D-Bnx,\quad 0=D-An,\ 0=-Bn,\quad C=0,\ D=\frac{1}{n}$

---

$$\int x^2\cos nx\,dx=\left[\begin{array}{l|l} u=x^2 & u'=2x\\ v'=\cos nx & v=\frac{\sin nx}{n}\end{array}\right]=\frac{x^2\sin nx}{n}-\frac{2}{n}\int x\sin nx\,dx$$
$$=\left[\begin{array}{l|l} u=x & u'=1\\ v'=\sin nx & v=-\frac{\cos nx}{n}\end{array}\right]=\frac{x^2\sin nx}{n}-\frac{2}{n}\left[-\frac{x\cos nx}{n}+\frac{1}{n}\int\cos nx\,dx\right]$$
$$=\frac{x^2\sin nx}{n}-\frac{2}{n}\left[-\frac{x\cos nx}{n}+\frac{1}{n}\,\frac{\sin nx}{n}\right]=\frac{2x\cos nx}{n^2}-\frac{2\sin nx}{n^3}+\frac{x^2\sin nx}{n}+c,$$
for $x\in R,\ n\in R-\{0\}$.

---

$$\int x^2\cos nx\,dx=(A+Bx+Cx^2)\cos nx+(D+Ex+Fx^2)\sin nx+c$$
$$=\frac{2x}{n^2}\cos nx+\left(-\frac{2}{n^3}+\frac{x^2}{n}\right)\sin nx+c,\ \text{for } x\in R,\ n\in R-\{0\}.$$

Derivative

$$x^2\cos nx=\left[\int x^2\cos nx\,dx\right]'=\left[(A+Bx+Cx^2)\cos nx+(D+Ex+Fx^2)\sin nx\right]'$$
$$=(B+2Cx)\cos nx-(A+Bx+Cx^2)n\sin nx+(E+2Fx)\sin nx+(D+Ex+Fx^2)n\cos nx$$
$$=[B+Dn+(2C+En)x+Fnx^2]\cos nx+[-An+E+(-Bn+2F)x-Cnx^2]\sin nx$$

Equations

$\cos nx:\ 0=B+Dn,\ 0=2C+En,\ 1=Fn,\quad A=0,\ B=\frac{2}{n^2},\ C=0$

$\sin nx:\ 0=-An+E,\ 0=-Bn+2F,\ 0=-Cn,\quad D=-\frac{2}{n^3},\ E=0,\ F=\frac{1}{n}$

---

$$\int x^3\cos nx\,dx=(A+Bx+Cx^2+Dx^3)\cos nx+(E+Fx+Gx^2+Hx^3)\sin nx+c$$
$$=\left(-\frac{6}{n^4}+\frac{3x^2}{n^2}\right)\cos nx+\left(-\frac{6x}{n^3}+\frac{x^3}{n}\right)\sin nx+c,\ \text{for } x\in R,\ n\in R-\{0\}.$$

Derivative

$$x^3\cos nx=\left[\int x^3\cos nx\,dx\right]'=\left[(A+Bx+Cx^2+Dx^3)\cos nx+(E+Fx+Gx^2+Hx^3)\sin nx\right]'$$
$$=(B+2Cx+3Dx^2)\cos nx-(A+Bx+Cx^2+Dx^3)n\sin nx$$
$$+(F+2Gx+3Hx^2)\sin nx+(E+Fx+Gx^2+Hx^3)n\cos nx$$
$$=[B+En+(2C+Fn)x-(3D+Gn)x^2+Hnx^3]\cos nx$$
$$+[-An+F+(-Bn+2G)x+(-Cn+3H)x^2-Dnx^3]\sin nx$$

Equations

$\cos nx:\ 0=B+En,\ 0=2C+Fn,\ 0=3D+Gn,\ 1=Hn,\quad A=-\frac{6}{n^4},\ B=D=0,\ C=\frac{3}{n^2}$

$\sin nx:\ 0=F-An,\ 0=2G-Bn,\ 0=3H-Cn,\ 0=-Dn,\quad E=G=0,\ F=-\frac{6}{n^3},\ H=\frac{1}{n}$

---

# 7 Integration and Series

$$\int e^x\,dx = \left[\, e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!} = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^k}{k!}+\cdots,\ x\in R \,\right] = \int\left[\sum_{k=0}^{\infty}\frac{x^k}{k!}\right]dx$$
$$= \sum_{k=0}^{\infty}\int\frac{x^k}{k!}\,dx = \sum_{k=0}^{\infty}\frac{x^{k+1}}{(k+1)k!}+c_1 = \sum_{k=0}^{\infty}\frac{x^{k+1}}{(k+1)!}+c_1 = \left[\begin{matrix} k+1=n \\ k=0,\ n=1\end{matrix}\right] = \sum_{n=1}^{\infty}\frac{x^n}{n!}+c_1$$
$$= \left[\, 1=\frac{x^0}{0!} \,\right] = \frac{x^0}{0!}+\sum_{n=1}^{\infty}\frac{x^n}{n!}+c_1-1 = \left[\, c_1-1=c \,\right] = \sum_{n=0}^{\infty}\frac{x^n}{n!}+c = e^x+c,\ \text{for } x\in R.$$

---

$$\int e^x\,dx = \int\left[1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^k}{k!}+\cdots\right]dx$$
$$= \int dx+\int x\,dx+\int\frac{x^2}{2!}\,dx+\int\frac{x^3}{3!}\,dx+\cdots+\int\frac{x^k}{k!}\,dx+\cdots$$
$$= x+\frac{x^2}{2}+\frac{x^3}{3\cdot 2!}+\frac{x^4}{4\cdot 3!}+\cdots+\frac{x^{k+1}}{(k+1)k!}+\cdots+c_1$$
$$= 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots+\frac{x^{k+1}}{(k+1)!}+\cdots+c_1-1 = e^x+c,\ \text{for } x\in R.$$

---

$$\int \sin x\,dx = \left[\, \sin x = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!} = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots+\frac{(-1)^k x^{2k+1}}{(2k+1)!}+\cdots,\ x\in R \,\right]$$
$$= \int\left[\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!}\right]dx = \sum_{k=0}^{\infty}\int\frac{(-1)^k x^{2k+1}}{(2k+1)!}\,dx = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+2}}{(2k+2)(2k+1)!}+c_1$$
$$= \left[\begin{matrix} k+1=n,\ k=0 \\ 2k+2=2n,\ n=1\end{matrix}\right] = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n}}{(2n)!}+c_1 = \left[\, 1=\frac{(-1)^0x^{2\cdot 0}}{(2\cdot 0)!} \,\right]$$
$$= -\frac{(-1)^0x^{2\cdot 0}}{(2\cdot 0)!}-\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}+c_1+1 = \left[\, c_1+1=c \,\right] = -\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}+c$$
$$= -\cos x+c,\ \text{for } x\in R.$$

---

$$\int \cos x\,dx = \left[\, \cos x = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!} = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots+\frac{(-1)^k x^{2k}}{(2k)!}+\cdots,\ x\in R \,\right]$$
$$= \int\left[\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!}\right]dx = \sum_{k=0}^{\infty}\int\frac{(-1)^k x^{2k}}{(2k)!}\,dx = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)(2k)!}+c = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!}+c$$
$$= \sin x+c,\ \text{for } x\in R.$$

---

$$\int e^{-x}\,dx = \left[\, e^{-x} = \sum_{k=0}^{\infty}\frac{(-x)^k}{k!} = \sum_{k=0}^{\infty}\frac{(-1)^k x^k}{k!} = 1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\cdots+\frac{(-1)^k x^k}{k!}+\cdots,\ x\in R \,\right]$$
$$= \int\left[\sum_{k=0}^{\infty}\frac{(-1)^k x^k}{k!}\right]dx = \sum_{k=0}^{\infty}\int\frac{(-1)^k x^k}{k!}\,dx = \sum_{k=0}^{\infty}\frac{(-1)^k x^{k+1}}{(k+1)k!}+c_1 = \left[\begin{matrix} k+1=n \\ k=0,\ n=1\end{matrix}\right]$$
$$= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n!}+c_1 = -\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{n!}+c_1 = \left[\, 1=\frac{x^0}{0!} \,\right] = -\frac{x^0}{0!}-\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{n!}+c_1+1$$
$$= \left[\, c_1+1=c \,\right] = -\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!}+c = -e^{-x}+c,\ \text{for } x\in R.$$

---

$$\int e^{(-x^2)}\,dx = \left[\, e^{(-x^2)} = \sum_{k=0}^{\infty}\frac{(-x^2)^k}{k!} = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{k!} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + \frac{(-1)^k x^{2k}}{k!} + \cdots,\quad x\in R \,\right]$$
$$= \int\left[\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{k!}\right]dx = \sum_{k=0}^{\infty}\int\frac{(-1)^k x^{2k}}{k!}\,dx = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)k!} + c,\ \text{for } x\in R.$$

---

$$\int e^{(x^2)}\,dx = \left[\, e^{(x^2)} = \sum_{k=0}^{\infty}\frac{(x^2)^k}{k!} = \sum_{k=0}^{\infty}\frac{x^{2k}}{k!} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots + \frac{x^{2k}}{k!} + \cdots,\quad x\in R \,\right]$$
$$= \int\left[\sum_{k=0}^{\infty}\frac{kx^{2k}}{k!}\right]dx = \sum_{k=0}^{\infty}\int\frac{x^{2k}}{k!}\,dx = \sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)k!} + c,\ \text{for } x\in R.$$

---

$$\int \sin(x^2)\,dx = \left[\, \sin(x^2) = \sum_{k=0}^{\infty}\frac{(-1)^k (x^2)^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty}\frac{(-1)^k x^{4k+2}}{(2k+1)!} = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots + \frac{(-1)^k x^{4k+2}}{(2k+1)!} + \cdots,\quad x\in R \,\right]$$
$$= \int\left[\sum_{k=0}^{\infty}\frac{(-1)^k x^{4k+2}}{(2k+1)!}\right]dx = \sum_{k=0}^{\infty}\int\frac{(-1)^k x^{4k+2}}{(2k+1)!}\,dx = \sum_{k=0}^{\infty}\frac{(-1)^k x^{4k+3}}{(4k+3)(2k+1)!} + c\ \text{for } x\in R.$$

---

$$\int \cos(x^2)\,dx = \left[\, \cos(x^2) = \sum_{k=0}^{\infty}\frac{(-1)^k (x^2)^{2k}}{(2k)!} = \sum_{k=0}^{\infty}\frac{(-1)^k x^{4k}}{(2k)!} = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \cdots + \frac{(-1)^k x^{4k}}{(2k)!} + \cdots,\quad x\in R \,\right]$$
$$= \int\left[\sum_{k=0}^{\infty}\frac{(-1)^k x^{4k}}{(2k)!}\right]dx = \sum_{k=0}^{\infty}\int\frac{(-1)^k x^{4k}}{(2k)!}\,dx = \sum_{k=0}^{\infty}\frac{(-1)^k x^{4k+1}}{(4k+1)(2k)!} + c,\ \text{for } x\in R.$$

---

$$\int \frac{e^x}{x}\,dx = \left[\, \frac{e^x}{x} = \frac{1}{x}\sum_{k=0}^{\infty}\frac{x^k}{k!} = \sum_{k=0}^{\infty}\frac{x^k}{x\cdot k!} = \frac{1}{x} + \sum_{k=1}^{\infty}\frac{x^{k-1}}{k!} = \frac{1}{x} + 1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots + \frac{x^{k-1}}{k!} + \cdots,\quad x\in R-\{0\} \,\right]$$
$$= \int\left[\frac{1}{x} + \sum_{k=1}^{\infty}\frac{x^{k-1}}{k!}\right]dx = \int\frac{dx}{x} + \sum_{k=1}^{\infty}\int\frac{x^{k-1}}{k!}\,dx = \ln|x| + \sum_{k=1}^{\infty}\frac{x^k}{k\cdot k!} + c,\ \text{for } x\in R-\{0\}.$$

---

$$\int \frac{\sin x}{x}\,dx = \left[\, \frac{\sin x}{x} = \frac{1}{x}\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{x(2k+1)!} = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k+1)!} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots + \frac{(-1)^k x^{2k}}{(2k+1)!} + \cdots,\quad x\in R-\{0\} \,\right]$$
$$= \int\left[\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k+1)!}\right]dx = \sum_{k=0}^{\infty}\int\frac{(-1)^k x^{2k}}{(2k+1)!}\,dx = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)(2k+1)!} + c,\ \text{for } x\in R-\{0\}.$$

---

$$\int \frac{\cos x}{x}\,dx = \left[\, \frac{\cos x}{x} = \frac{1}{x}\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!} = \frac{(-1)^0 x^{2\cdot 0}}{x(2\cdot 0)!} + \sum_{k=1}^{\infty}\frac{(-1)^k x^{2k}}{x(2k)!} = \frac{1}{x} + \sum_{k=1}^{\infty}\frac{(-1)^k x^{2k-1}}{(2k)!} = \frac{1}{x} - \frac{x}{2!} + \frac{x^3}{4!} - \frac{x^5}{6!} + \cdots + \frac{(-1)^k x^{2k-1}}{(2k)!} + \cdots,\quad x\in R-\{0\} \,\right]$$
$$= \int\left[\frac{1}{x} + \sum_{k=1}^{\infty}\frac{(-1)^k x^{2k-1}}{(2k)!}\right]dx = \int\frac{dx}{x} + \sum_{k=1}^{\infty}\int\frac{(-1)^k x^{2k-1}}{(2k)!}\,dx = \ln|x| + \sum_{k=1}^{\infty}\frac{(-1)^k x^{2k}}{2k(2k)!},$$
for $x\in R-\{0\}$.

# 8 Fourier Series

$\int_a^a f(x)\,\mathrm{d}x = 0$ for all $a \in R$. $\int_a^b f(x) = -\int_b^a f(x)$, for all $a, b \in R$, $f(x) \in R_{\langle a\,;\,b\rangle}$.

$f(x) \in R_{\langle a\,;\,b\rangle}$ ... $f(x)$ is Riemann integrable on $\langle a\,;\,b\rangle$ ... exists finite $\int_a^b f(x) \in R$.

**Theorem.**

$$\int_a^{a+2\pi} f(x)\,\mathrm{d}x = \int_0^{2\pi} f(x)\,\mathrm{d}x = \int_{-\pi}^{\pi} f(x)\,\mathrm{d}x,$$

for a periodic function $f(x) \in R_{\langle a\,;\,a+2\pi\rangle}$ with period $2\pi$, $a > 0$.

$$\int_a^{a+2\pi} f(x)\,\mathrm{d}x = \int_a^{2\pi} f(x)\,\mathrm{d}x + \int_{2\pi}^{a+2\pi} f(x)\,\mathrm{d}x = \left[\begin{array}{c|c|c} x = t-2\pi, \quad \mathrm{d}x = \mathrm{d}t & x \to 2\pi & t \to 0 \\ x \in \langle 2\pi\,;\,a+2\pi\rangle\,,\ t \in \langle 0\,;\,a\rangle & x \to a+2\pi & t \to a \end{array}\right]$$
$$= \int_a^{2\pi} f(x)\,\mathrm{d}x + \int_0^a f(t-2\pi)\,\mathrm{d}t = \left[\begin{array}{c} \text{periodic function} \\ f(t-2\pi) = f(t) \end{array}\right] = \int_a^{2\pi} f(x)\,\mathrm{d}x + \int_0^a f(t)\,\mathrm{d}t$$
$$= \int_a^{2\pi} f(x)\,\mathrm{d}x + \int_0^a f(x)\,\mathrm{d}x = \int_0^{2\pi} f(x)\,\mathrm{d}x = \int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \int_{-2\pi}^{0} f(x)\,\mathrm{d}x,$$

for $a \in R$, $x \in D(f)$, $f(x) \in R_{\langle 0\,;\,2\pi\rangle}$ is a periodic with period $2\pi$.

**Theorem.**

$$\int_a^b f(x)\,\mathrm{d}x = \int_{-b}^{-a} f(-x)\,\mathrm{d}x, \quad \text{for } f(x) \in R_{\langle a\,;\,b\rangle},\ a, b \in R,\ a < b,$$

specially

$$\int_a^b f(x)\,\mathrm{d}x = \int_{-b}^{-a} f(-x)\,\mathrm{d}x = -\int_{-b}^{-a} f(x)\,\mathrm{d}x, \quad \text{for a odd function } f(x) \in R_{\langle a\,;\,b\rangle},$$

$$\int_a^b f(x)\,\mathrm{d}x = \int_{-b}^{-a} f(-x)\,\mathrm{d}x = \int_{-b}^{-a} f(x)\,\mathrm{d}x, \quad \text{for a even function } f(x) \in R_{\langle a\,;\,b\rangle},$$

$$\int_{-a}^{a} f(x)\,\mathrm{d}x = 0, \quad \text{for a odd function } f(x) \in R_{\langle -a\,;\,a\rangle},\ a > 0,$$

$$\int_{-a}^{a} f(x)\,\mathrm{d}x = 2\int_0^a f(x)\,\mathrm{d}x, \quad \text{for a even function } f(x) \in R_{\langle -a\,;\,a\rangle},\ a > 0.$$

$$\int_a^b f(x)\,\mathrm{d}x = \left[\begin{array}{c|c|c} x=-t,\ x\in\langle a\,;\,b\rangle & x\to a & t\to -a \\ \mathrm{d}x=-\,\mathrm{d}t,\ t\in\langle -b\,;\,-a\rangle & x\to b & t\to -b \end{array}\right] = -\int_{-a}^{-b} f(-t)\,\mathrm{d}t = \int_{-b}^{-a} f(-t)\,\mathrm{d}t$$
$$= \int_{-b}^{-a} f(-x)\,\mathrm{d}x, \text{ for } f(x)\in R_{\langle a\,;\,b\rangle},\ a,b\in R,\ a<b,$$

$$\int_a^b f(x)\,\mathrm{d}x = \begin{cases} \left[\, f(x)=f(-x)\,\right] = \displaystyle\int_{-b}^{-a} f(x)\,\mathrm{d}x, \text{ for a even function } f(x)\in R_{\langle a\,;\,b\rangle}, \\ \left[\, f(x)=-f(-x)\,\right] = -\displaystyle\int_{-b}^{-a} f(x)\,\mathrm{d}x, \text{ for a odd function } f(x)\in R_{\langle a\,;\,b\rangle}. \end{cases}$$

$$\int_{-a}^a f(x)\,\mathrm{d}x = \int_{-a}^0 f(x)\,\mathrm{d}x + \int_0^a f(x)\,\mathrm{d}x = \begin{cases} 2\displaystyle\int_0^a f(x)\,\mathrm{d}x, \text{ for a even } f(x)\in R_{\langle -a\,;\,a\rangle}, \\ 0, \text{ for a odd } f(x)\in R_{\langle -a\,;\,a\rangle},\ a>0. \end{cases}$$

---

$$\int_a^{a+2\pi} \cos nx\,\mathrm{d}x = \int_{-\pi}^{\pi} \cos nx\,\mathrm{d}x = \int_0^{2\pi} \cos nx\,\mathrm{d}x = \left[\frac{\sin nx}{n}\right]_a^{a+2\pi} = \frac{\sin(na+2n\pi)}{n} - \frac{\sin na}{n}$$
$$= \left[\begin{array}{c} \text{periodic function} \\ \sin(na+2n\pi)=\sin na \end{array}\right] = \frac{\sin na}{n} - \frac{\sin na}{n} = 0, \text{ for } n\in N,\ a\in R.$$

---

$$\int_a^{a+2\pi} \sin nx\,\mathrm{d}x = \int_{-\pi}^{\pi} \sin nx\,\mathrm{d}x = \int_0^{2\pi} \sin nx\,\mathrm{d}x = \left[-\frac{\cos nx}{n}\right]_a^{a+2\pi} = -\frac{\cos(na+2n\pi)}{n} + \frac{\cos na}{n}$$
$$= \left[\begin{array}{c} \text{periodic function} \\ \cos(na+2n\pi)=\cos na \end{array}\right] = -\frac{\cos na}{n} + \frac{\cos na}{n} = 0, \text{ for } n\in N,\ a\in R.$$

---

$$\int_a^{a+2\pi} \cos nx\cdot\sin mx\,\mathrm{d}x = \int_{-\pi}^{\pi} \cos nx\cdot\sin mx\,\mathrm{d}x = \int_0^{2\pi} \cos nx\cdot\sin mx\,\mathrm{d}x = 0,$$
for $m,n\in N,\ a\in R$.

---

$$I = \int_0^{2\pi} \cos nx\cdot\sin mx\,\mathrm{d}x = \left[\begin{array}{l|l} u=\sin mx & u'=m\cos mx \\ v'=\cos nx & v=\frac{1}{n}\sin nx \end{array}\right]$$
$$= \left[\frac{\sin nx\cdot\sin mx}{n}\right]_0^{2\pi} - \frac{m}{n}\int_0^{2\pi} \sin nx\cdot\cos mx\,\mathrm{d}x = \left[\begin{array}{l} \sin 2n\pi=0,\ \sin 0=0 \\ \sin 2m\pi=0,\ \sin 0=0 \end{array}\right]$$
$$= \left[\frac{0\cdot 0}{n} - \frac{0\cdot 0}{n}\right] - \frac{m}{n}\int_0^{2\pi} \sin nx\cdot\cos mx\,\mathrm{d}x = -\frac{m}{n}\int_0^{2\pi} \sin nx\cdot\cos mx\,\mathrm{d}x$$
$$= \left[\begin{array}{l|l} u=\cos mx & u'=-m\sin mx \\ v'=\sin nx & v=-\frac{1}{n}\cos nx \end{array}\right] = -\frac{m}{n}\left[-\frac{\cos nx\cdot\cos mx}{n}\right]_0^{2\pi} + \frac{m}{n}\frac{m}{n}\int_0^{2\pi} \cos nx\cdot\sin mx\,\mathrm{d}x$$
$$= \left[\begin{array}{l} \cos 2n\pi=1,\ \cos 0=1 \\ \cos 2m\pi=1,\ \cos 0=1 \end{array}\right] = -\frac{m}{n}\left[-\frac{1\cdot 1}{n} + \frac{1\cdot 1}{n}\right] + \frac{m^2}{n^2}\int_0^{2\pi} \cos nx\cdot\sin mx\,\mathrm{d}x$$
$$= \frac{m^2}{n^2}\int_0^{2\pi} \cos nx\cdot\sin mx\,\mathrm{d}x = \frac{m^2}{n^2} I = \left[\begin{array}{l} \boxed{\text{Equation}}\quad I=\frac{m^2}{n^2}I,\ \left(1-\frac{m^2}{n^2}\right)I=0 \\ m\neq n,\ 1\neq\frac{m^2}{n^2},\ 1-\frac{m^2}{n^2}\neq 0\ \Rightarrow\ I=0 \end{array}\right] = 0,$$
for $m,n\in N,\ m\neq n$.

---

$$\int_0^{2\pi} \cos nx \cdot \sin mx \,\mathrm{d}x = \left[ \cos\alpha \sin\beta = \tfrac{\sin(\beta-\alpha)+\sin(\beta+\alpha)}{2} \right] = \int_0^{2\pi} \frac{\sin(mx-nx)+\sin(mx+nx)}{2} \,\mathrm{d}x$$
$$= \int_0^{2\pi} \left[ \frac{\sin(m-n)x}{2} + \frac{\sin(m+n)x}{2} \right] \mathrm{d}x = \left[ -\frac{\cos(m-n)x}{2(m-n)} - \frac{\cos(m+n)x}{2(m+n)} \right]_0^{2\pi}$$
$$= -\frac{\cos 2(m-n)\pi}{2(m-n)} - \frac{\cos 2(m+n)\pi}{2(m+n)} + \frac{\cos 0}{2(m-n)} + \frac{\cos 0}{2(m+n)}$$
$$= \left[ \cos 2(m-n)\pi = 1,\ \cos 2(m+n)\pi = 1,\ \cos 0 = 1 \right]$$
$$= -\frac{1}{2(m-n)} - \frac{1}{2(m+n)} + \frac{1}{2(m-n)} + \frac{1}{2(m+n)} = 0, \text{ for } m,n \in N,\ m \neq n.$$

---

$$\int_0^{2\pi} \cos nx \cdot \sin nx \,\mathrm{d}x = \left[ \cos\alpha \sin\alpha = \tfrac{\sin 2\alpha}{2} \right] = \frac{1}{2} \int_0^{2\pi} \sin 2nx \,\mathrm{d}x = \frac{1}{2} \left[ -\frac{\cos 2nx}{2n} \right]_0^{2\pi}$$
$$= \frac{1}{2} \left[ -\frac{\cos 4n\pi}{2n} + \frac{\cos 0}{2n} \right] = \frac{1}{2} \left[ -\frac{1}{2n} + \frac{1}{2n} \right] = 0, \text{ for } m = n.$$

---

$$\int_a^{a+2\pi} \cos nx \cdot \cos mx \,\mathrm{d}x = \int_{-\pi}^{\pi} \cos nx \cdot \cos mx \,\mathrm{d}x = \int_0^{2\pi} \cos nx \cdot \cos mx \,\mathrm{d}x = 0,$$
for $m, n \in N,\ m \neq n,\ a \in R$.

---

$$I = \int_0^{2\pi} \cos nx \cdot \cos mx \,\mathrm{d}x = \left[ \begin{array}{l|l} u = \cos mx & u' = -m \sin mx \\ v' = \cos nx & v = \frac{1}{n} \sin nx \end{array} \right]$$
$$= \left[ \frac{\sin nx \cdot \cos mx}{n} \right]_0^{2\pi} + \frac{m}{n} \int_0^{2\pi} \sin nx \cdot \sin mx \,\mathrm{d}x = \left[ \begin{array}{l} \sin 2n\pi = 0,\ \sin 0 = 0 \\ \cos 2m\pi = 1,\ \cos 0 = 1 \end{array} \right]$$
$$= \left[ \frac{0 \cdot 1}{n} - \frac{0 \cdot 1}{n} \right] + \frac{m}{n} \int_0^{2\pi} \sin nx \cdot \sin mx \,\mathrm{d}x = \frac{m}{n} \int_0^{2\pi} \sin nx \cdot \sin mx \,\mathrm{d}x$$
$$= \left[ \begin{array}{l|l} u = \sin mx & u' = m \cos mx \\ v' = \sin nx & v = -\frac{1}{n} \cos nx \end{array} \right] = \frac{m}{n} \left[ -\frac{\cos nx \cdot \sin mx}{n} \right]_0^{2\pi} + \frac{m}{n} \frac{m}{n} \int_0^{2\pi} \cos nx \cdot \cos mx \,\mathrm{d}x$$
$$= \left[ \begin{array}{l} \cos 2n\pi = 1,\ \cos 0 = 1 \\ \sin 2m\pi = 0,\ \sin 0 = 0 \end{array} \right] = \frac{m}{n} \left[ -\frac{1 \cdot 0}{n} + \frac{1 \cdot 0}{n} \right] + \frac{m^2}{n^2} \int_0^{2\pi} \cos nx \cdot \cos mx \,\mathrm{d}x$$
$$= \frac{m^2}{n^2} \int_0^{2\pi} \cos nx \cdot \cos mx \,\mathrm{d}x = \frac{m^2}{n^2} I = \left[ \begin{array}{l} \boxed{\text{Equation}} \quad I = \frac{m^2}{n^2} I,\ \left(1 - \frac{m^2}{n^2}\right) I = 0 \\ m \neq n,\ 1 \neq \frac{m^2}{n^2},\ 1 - \frac{m^2}{n^2} \neq 0 \ \Rightarrow\ I = 0 \end{array} \right] = 0,$$
for $m, n \in N,\ m \neq n$.

---

$$\int_0^{2\pi} \cos nx \cdot \cos mx \,\mathrm{d}x = \left[ \cos\alpha \cos\beta = \tfrac{\cos(\alpha+\beta)+\cos(\alpha-\beta)}{2} \right] = \int_0^{2\pi} \frac{\cos(nx+mx)+\cos(nx-mx)}{2} \,\mathrm{d}x$$
$$= \int_0^{2\pi} \left[ \frac{\cos(n+m)x}{2} + \frac{\cos(n-m)x}{2} \right] \mathrm{d}x = \left[ \frac{\sin(n+m)x}{2(n+m)} + \frac{\sin(n-m)x}{2(n-m)} \right]_0^{2\pi}$$
$$= \frac{\sin 2(m-n)\pi}{2(m-n)} + \frac{\sin 2(m+n)\pi}{2(m+n)} - \frac{\sin 0}{2(m-n)} + \frac{\sin 0}{2(m+n)}$$
$$= \left[ \sin 2(m-n)\pi = 0,\ \sin 2(m+n)\pi = 0,\ \sin 0 = 0 \right]$$
$$= \frac{0}{2(m-n)} + \frac{0}{2(m+n)} - \frac{0}{2(m-n)} - \frac{0}{2(m+n)} = 0, \text{ for } m,n \in N,\ m \neq n,\ x \in R.$$

---

$$\int_a^{a+2\pi} \cos^2 nx \, dx = \int_{-\pi}^{\pi} \cos^2 nx \, dx = \int_0^{2\pi} \cos^2 nx \, dx = \pi, \text{ for } n \in N,\ a \in R.$$

---

$$\int_0^{2\pi} \cos^2 nx \, dx = \left[ \cos^2\alpha = \tfrac{1+\cos 2\alpha}{2} \right] = \int_0^{2\pi} \frac{1+\cos 2nx}{2} \, dx = \int_0^{2\pi} \left[\tfrac{1}{2} + \tfrac{\cos 2nx}{2}\right] dx$$
$$= \left[\frac{x}{2} + \frac{\sin 2nx}{2\cdot 2n}\right]_0^{2\pi} = \left[\begin{array}{c} \sin 4n\pi = 0 \\ \sin 0 = 0 \end{array}\right] = \frac{2\pi}{2} + \frac{\sin 4n\pi}{4n} - \frac{0}{2} - \frac{\sin 0}{4n} = \pi, \text{ for } n \in N.$$

---

$$\int_a^{a+2\pi} \sin nx \cdot \sin mx \, dx = \int_{-\pi}^{\pi} \sin nx \cdot \sin mx \, dx = \int_0^{2\pi} \sin nx \cdot \sin mx \, dx = 0,$$
for $m, n \in N,\ m \neq n,\ a \in R$.

---

$$I = \int_0^{2\pi} \sin nx \cdot \sin mx \, dx = \left[\begin{array}{c|c} u = \sin mx & u' = m\cos mx \\ v' = \sin nx & v = -\frac{1}{n}\cos nx \end{array}\right]$$
$$= \left[-\frac{\cos nx \cdot \sin mx}{n}\right]_0^{2\pi} + \frac{m}{n}\int_0^{2\pi} \cos nx \cdot \cos mx \, dx = \left[\begin{array}{c} \cos 2n\pi = 1,\ \cos 0 = 1 \\ \sin 2m\pi = 0,\ \sin 0 = 0 \end{array}\right]$$
$$= \left[\frac{1\cdot 0}{n} - \frac{1\cdot 0}{n}\right] + \frac{m}{n}\int_0^{2\pi} \cos nx \cdot \cos mx \, dx = \frac{m}{n}\int_0^{2\pi} \cos nx \cdot \cos mx \, dx$$
$$= \left[\begin{array}{c|c} u = \cos mx & u' = -m\sin mx \\ v' = \cos nx & v = \frac{1}{n}\sin nx \end{array}\right] = \frac{m}{n}\left[\frac{\sin nx \cdot \cos mx}{n}\right]_0^{2\pi} + \frac{m}{n}\frac{m}{n}\int_0^{2\pi} \sin nx \cdot \sin mx \, dx$$
$$= \left[\begin{array}{c} \sin 2n\pi = 0,\ \sin 0 = 0 \\ \cos 2m\pi = 1,\ \cos 0 = 1 \end{array}\right] = \frac{m}{n}\left[\frac{0\cdot 1}{n} - \frac{0\cdot 1}{n}\right] + \frac{m^2}{n^2}\int_0^{2\pi} \sin nx \cdot \sin mx \, dx$$
$$= \frac{m^2}{n^2}\int_0^{2\pi} \sin nx \cdot \sin mx \, dx = \frac{m^2}{n^2} I = \left[\begin{array}{c} \boxed{\text{Equation}}\quad I = \frac{m^2}{n^2} I,\ \left(1 - \frac{m^2}{n^2}\right) I = 0 \\ m \neq n,\ 1 \neq \frac{m^2}{n^2},\ 1 - \frac{m^2}{n^2} \neq 0 \ \Rightarrow\ I = 0 \end{array}\right] = 0,$$
for $m, n \in N,\ m \neq n$.

---

$$\int_0^{2\pi} \sin nx \cdot \sin mx \, dx = \left[ \sin\alpha\sin\beta = \tfrac{\cos(\alpha-\beta) - \cos(\alpha+\beta)}{2} \right] = \int_0^{2\pi} \frac{\cos(nx-mx) - \cos(nx+mx)}{2} \, dx$$
$$= \int_0^{2\pi} \left[\frac{\cos(n-m)x}{2} - \frac{\cos(n+m)x}{2}\right] dx = \left[\frac{\sin(n-m)x}{2(n-m)} - \frac{\sin(n+m)x}{2(n+m)}\right]_0^{2\pi}$$
$$= \frac{\sin 2(n-m)\pi}{2(n-m)} - \frac{\sin 2(n+m)\pi}{2(n+m)} + \frac{\sin 0}{2(n-m)} - \frac{\sin 0}{2(n+m)} = \left[\begin{array}{c} \sin 2(n-m)\pi = 0 \\ \sin 2(n+m)\pi = 0,\ \sin 0 = 0 \end{array}\right]$$
$$= \frac{0}{2(n-m)} + \frac{0}{2(n+m)} - \frac{0}{2(n-m)} - \frac{0}{2(n+m)} = 0, \text{ for } m, n \in N,\ m \neq n,\ x \in R.$$

---

$$\int_a^{a+2\pi} \sin^2 nx \, dx = \int_{-\pi}^{\pi} \sin^2 nx \, dx = \int_0^{2\pi} \sin^2 nx \, dx = \pi, \text{ for } n \in N,\ a \in R.$$

---

$$\int_0^{2\pi} \sin^2 nx \, dx = \left[ \sin^2\alpha = \tfrac{1-\cos 2\alpha}{2} \right] = \int_0^{2\pi} \frac{1-\cos 2nx}{2} \, dx = \int_0^{2\pi} \left[\tfrac{1}{2} - \tfrac{\cos 2nx}{2}\right] dx$$
$$= \left[\frac{x}{2} - \frac{\sin 2nx}{2\cdot 2n}\right]_0^{2\pi} = \left[\begin{array}{c} \sin 4n\pi = 0 \\ \sin 0 = 0 \end{array}\right] = \frac{2\pi}{2} - \frac{\sin 4n\pi}{4n} - \frac{0}{2} + \frac{\sin 0}{4n} = \pi, \text{ for } n \in N.$$

---

## Scalar Product and Norm in $R_{\langle a\,;\,a+2\pi\rangle}$

$$(f,g)=\int_a^{a+2\pi} f(x)g(x)\,\mathrm{d}x,\quad \|f\|=\sqrt{(f,f)}=\sqrt{\int_a^{a+2\pi} f^2(x)\,\mathrm{d}x},\quad \text{for } f,g\in R_{\langle a\,;\,a+2\pi\rangle}.$$

## Scalar Product and Norm in $R_{\langle a\,;\,a+2l\rangle}$

$$(f,g)=\int_a^{a+2l} f(x)g(x)\,\mathrm{d}x,\quad \|f\|=\sqrt{(f,f)}=\sqrt{\int_a^{a+2l} f^2(x)\,\mathrm{d}x},\quad \text{for } f,g\in R_{\langle a\,;\,a+2l\rangle}.$$

---

$$(1,1)=\int_a^{a+2\pi} 1^2\,\mathrm{d}x=\int_a^{a+2\pi}\mathrm{d}x=\Big[x\Big]_a^{a+2\pi}=a+2\pi-a=2\pi,$$

$$\|1\|=\sqrt{(1,1)}=\sqrt{2\pi},\ \text{for } a\in R.$$

---

$$(1,\cos nx)=\int_a^{a+2\pi}\cos nx\,\mathrm{d}x=\big[\ \text{p. 64}\ \big]=0,\ \text{for } n\in N,\ a\in R.$$

---

$$(1,\sin nx)=\int_a^{a+2\pi}\sin nx\,\mathrm{d}x=\big[\ \text{p. 64}\ \big]=0,\ \text{for } n\in N,\ a\in R.$$

---

$$(\cos nx,\sin mx)=\int_a^{a+2\pi}\cos nx\cdot\sin mx\,\mathrm{d}x=\big[\ \text{p. 64}\ \big]=0,\ \text{for } m,n\in N,\ a\in R.$$

---

$$(\cos nx,\cos mx)=\int_a^{a+2\pi}\cos nx\cdot\cos mx\,\mathrm{d}x=\big[\ \text{p. 65}\ \big]=0,\ \text{for } m,n\in N,\ m\neq n,\ a\in R.$$

---

$$(\cos nx,\cos nx)=\int_a^{a+2\pi}\cos^2 nx\,\mathrm{d}x=\big[\ \text{p. 66}\ \big]=\pi,$$

$$\|\cos nx\|=\sqrt{(\cos nx,\cos nx)}=\sqrt{\pi},\ \text{for } n\in N,\ a\in R.$$

---

$$(\sin nx,\sin mx)=\int_a^{a+2\pi}\sin nx\cdot\sin mx\,\mathrm{d}x=\big[\ \text{p. 66}\ \big]=0,\ \text{for } m,n\in N,\ m\neq n,\ a\in R.$$

---

$$(\sin nx,\sin nx)=\int_a^{a+2\pi}\sin^2 nx\,\mathrm{d}x=\big[\ \text{p. 66}\ \big]=\pi,$$

$$\|\sin nx\|=\sqrt{(\sin nx,\sin nx)}=\sqrt{\pi},\ \text{for } n\in N,\ a\in R.$$

---

$$(1,1) = \int_a^{a+2l} 1^2\,\mathrm{d}x = \int_a^{a+2l} \mathrm{d}x = \Big[x\Big]_a^{a+2l} = a+2l-a = 2l,$$
$$\|1\| = \sqrt{(1,1)} = \sqrt{2l},\ \text{for } a\in R,\ l>0.$$

---

$$(1, \cos\tfrac{n\pi x}{l}) = \int_a^{a+2l} \cos\tfrac{n\pi x}{l}\,\mathrm{d}x = \left[\begin{array}{l|l|l} t=\frac{\pi x}{l},\ x=\frac{lt}{\pi},\ x\in\langle a;\, a+2l\rangle & x\to a & x\to a+2l \\ \mathrm{d}x=\frac{l}{\pi}\,\mathrm{d}t,\ t\in\langle\frac{\pi a}{l};\, \frac{\pi a}{l}+2\pi\rangle & t\to\frac{\pi a}{l} & t\to\frac{\pi a}{l}+2\pi \end{array}\right]$$
$$= \tfrac{l}{\pi}\int_{\frac{\pi a}{l}}^{\frac{\pi a}{l}+2\pi} \cos nt\,\mathrm{d}t = \big[\text{ p. 64 }\big] = \tfrac{l}{\pi}\cdot 0 = 0,\ \text{for } n\in N,\ a\in R,\ l>0.$$

---

$$(1, \sin\tfrac{n\pi x}{l}) = \int_a^{a+2l} \sin\tfrac{n\pi x}{l}\,\mathrm{d}x = \left[\begin{array}{l|l|l} t=\frac{\pi x}{l},\ x=\frac{lt}{\pi},\ x\in\langle a;\, a+2l\rangle & x\to a & x\to a+2l \\ \mathrm{d}x=\frac{l}{\pi}\,\mathrm{d}t,\ t\in\langle\frac{\pi a}{l};\, \frac{\pi a}{l}+2\pi\rangle & t\to\frac{\pi a}{l} & t\to\frac{\pi a}{l}+2\pi \end{array}\right]$$
$$= \tfrac{l}{\pi}\int_{\frac{\pi a}{l}}^{\frac{\pi a}{l}+2\pi} \sin nt\,\mathrm{d}t = \big[\text{ p. 64 }\big] = \tfrac{l}{\pi}\cdot 0 = 0,\ \text{for } n\in N,\ a\in R,\ l>0.$$

---

$$(\cos\tfrac{n\pi x}{l}, \sin\tfrac{m\pi x}{l}) = \int_a^{a+2l} \cos\tfrac{n\pi x}{l}\cdot\sin\tfrac{m\pi x}{l}\,\mathrm{d}x$$
$$= \left[\begin{array}{l|l|l} t=\frac{\pi x}{l},\ x=\frac{lt}{\pi},\ x\in\langle a;\, a+2l\rangle & x\to a & x\to a+2l \\ \mathrm{d}x=\frac{l}{\pi}\,\mathrm{d}t,\ t\in\langle\frac{\pi a}{l};\, \frac{\pi a}{l}+2\pi\rangle & t\to\frac{\pi a}{l} & t\to\frac{\pi a}{l}+2\pi \end{array}\right] = \tfrac{l}{\pi}\int_{\frac{\pi a}{l}}^{\frac{\pi a}{l}+2\pi} \cos nt\cdot\sin mt\,\mathrm{d}t$$
$$= \big[\text{ p. 64 }\big] = \tfrac{l}{\pi}\cdot 0 = 0,\ \text{for } m,n\in N,\ a\in R,\ l>0.$$

---

$$(\cos\tfrac{n\pi x}{l}, \cos\tfrac{m\pi x}{l}) = \int_a^{a+2l} \cos\tfrac{n\pi x}{l}\cdot\cos\tfrac{m\pi x}{l}\,\mathrm{d}x$$
$$= \left[\begin{array}{l|l|l} t=\frac{\pi x}{l},\ x=\frac{lt}{\pi},\ x\in\langle a;\, a+2l\rangle & x\to a & x\to a+2l \\ \mathrm{d}x=\frac{l}{\pi}\,\mathrm{d}t,\ t\in\langle\frac{\pi a}{l};\, \frac{\pi a}{l}+2\pi\rangle & t\to\frac{\pi a}{l} & t\to\frac{\pi a}{l}+2\pi \end{array}\right] = \tfrac{l}{\pi}\int_{\frac{\pi a}{l}}^{\frac{\pi a}{l}+2\pi} \cos nt\cdot\cos mt\,\mathrm{d}t$$
$$= \big[\text{ p. 65 }\big] = \tfrac{l}{\pi}\cdot 0 = 0,\ \text{for } m,n\in N,\ m\neq n,\ a\in R,\ l>0.$$

---

$$(\cos\tfrac{n\pi x}{l}, \cos\tfrac{n\pi x}{l}) = \int_a^{a+2l} \cos^2\tfrac{n\pi x}{l}\,\mathrm{d}x$$
$$= \left[\begin{array}{l|l|l} t=\frac{\pi x}{l},\ x=\frac{lt}{\pi},\ x\in\langle a;\, a+2l\rangle & x\to a & x\to a+2l \\ \mathrm{d}x=\frac{l}{\pi}\,\mathrm{d}t,\ t\in\langle\frac{\pi a}{l};\, \frac{\pi a}{l}+2\pi\rangle & t\to\frac{\pi a}{l} & t\to\frac{\pi a}{l}+2\pi \end{array}\right] = \tfrac{l}{\pi}\int_{\frac{\pi a}{l}}^{\frac{\pi a}{l}+2\pi} \cos^2 nt\,\mathrm{d}t$$
$$= \big[\text{ p. 66 }\big] = \tfrac{l}{\pi}\cdot\pi = l,$$
$$\|\cos\tfrac{n\pi x}{l}\| = \sqrt{(\cos\tfrac{n\pi x}{l}, \cos\tfrac{n\pi x}{l})} = \sqrt{l},\ \text{for } n\in N,\ a\in R,\ l>0.$$

---

$$(\sin\tfrac{n\pi x}{l}, \sin\tfrac{m\pi x}{l}) = \int_a^{a+2l} \sin\tfrac{n\pi x}{l}\cdot\sin\tfrac{m\pi x}{l}\,\mathrm{d}x$$
$$= \left[\begin{array}{l|l|l} t=\frac{\pi x}{l},\ x=\frac{lt}{\pi},\ x\in\langle a;\, a+2l\rangle & x\to a & x\to a+2l \\ \mathrm{d}x=\frac{l}{\pi}\,\mathrm{d}t,\ t\in\langle\frac{\pi a}{l};\, \frac{\pi a}{l}+2\pi\rangle & t\to\frac{\pi a}{l} & t\to\frac{\pi a}{l}+2\pi \end{array}\right] = \tfrac{l}{\pi}\int_{\frac{\pi a}{l}}^{\frac{\pi a}{l}+2\pi} \sin nt\cdot\sin mt\,\mathrm{d}t$$
$$= \big[\text{ p. 66 }\big] = \tfrac{l}{\pi}\cdot 0 = 0,\ \text{for } m,n\in N,\ m\neq n,\ a\in R,\ l>0.$$

---

$$(\sin\tfrac{n\pi x}{l},\sin\tfrac{n\pi x}{l}) = \int_a^{a+2l}\sin^2\tfrac{n\pi x}{l}\,\mathrm{d}x$$
$$= \left[\begin{array}{c} t=\frac{\pi x}{l},\ x=\frac{lt}{\pi},\ x\in\langle a;\,a+2l\rangle \\ \mathrm{d}x=\frac{l}{\pi}\,\mathrm{d}t,\ t\in\langle\frac{\pi a}{l};\,\frac{\pi a}{l}+2\pi\rangle \end{array}\middle|\begin{array}{c} x\to a \\ t\to\frac{\pi a}{l}\end{array}\middle|\begin{array}{c} x\to a+2l \\ t\to\frac{\pi a}{l}+2\pi\end{array}\right] = \frac{l}{\pi}\int_{\frac{\pi a}{l}}^{\frac{\pi a}{l}+2\pi}\sin^2 nt\,\mathrm{d}t$$
$$= \left[\text{ p. 66 }\right] = \tfrac{l}{\pi}\cdot\pi = l,$$
$$\|\sin\tfrac{n\pi x}{l}\| = \sqrt{(\sin\tfrac{n\pi x}{l},\sin\tfrac{n\pi x}{l})} = \sqrt{l},\ \text{for } n\in N,\ a\in R,\ l>0.$$

**Theorem.**

The **Fourier series** of a function $f(x)\in R_{\langle a\,;\,a+2\pi\rangle}$, $a\in R$ is given by
$$f(x)\approx A_0\tfrac{1}{\sqrt{2\pi}}+\sum_{n=1}^{\infty}\left[A_n\tfrac{\cos nx}{\sqrt{\pi}}+B_n\tfrac{\sin nx}{\sqrt{\pi}}\right],\quad \text{for } x\in\langle a\,;\,a+2\pi\rangle,$$
where
$$A_0=\tfrac{1}{\sqrt{2\pi}}\int_a^{a+2\pi}f(x)\,\mathrm{d}x,\ \ A_n=\tfrac{1}{\sqrt{2\pi}}\int_a^{a+2\pi}f(x)\cos nx\,\mathrm{d}x,\ \ B_n=\tfrac{1}{\sqrt{2\pi}}\int_a^{a+2\pi}f(x)\sin nx\,\mathrm{d}x,\ \ n\in N,$$
respectively
$$f(x)\approx\tfrac{a_0}{2}+\sum_{n=1}^{\infty}[a_n\cos nx+b_n\sin nx],\quad \text{for } x\in\langle a\,;\,a+2\pi\rangle,$$
where
$$a_0=\tfrac{1}{\pi}\int_a^{a+2\pi}f(x)\,\mathrm{d}x,\ \ a_n=\tfrac{1}{\pi}\int_a^{a+2\pi}f(x)\cos nx\,\mathrm{d}x,\ \ b_n=\tfrac{1}{\pi}\int_a^{a+2\pi}f(x)\sin nx\,\mathrm{d}x,\ \ n\in N.$$

A Fourier series converges to the function $\tilde{f}(x)$, $x\in R$:
$$\tilde{f}(x_0)=\begin{cases}\frac{1}{2}\Big[\lim\limits_{x\to x_0^-}f(x)+\lim\limits_{x\to x_0^+}f(x)\Big], & x_0\in(a\,;\,a+2\pi),\\ f(x_0-2k\pi), & x_0\in(a+2k\pi\,;\,a+2\pi+2k\pi),\ k\in Z,\\ \frac{1}{2}\Big[\lim\limits_{x\to a^+}f(x)+\lim\limits_{x\to a+2\pi^-}f(x)\Big], & x_0=a+2k\pi,\ k\in Z.\end{cases}$$

Function $\tilde{f}(x)$ is a periodic function with period $2\pi$ and equals to the original function at the interval $\langle a\,;\,a+2\pi\rangle$ at points of continuity or to the average of the two limits at points of discontinuity.

$$f(x)\approx\tilde{f}(x)=A_0\tfrac{1}{\sqrt{2\pi}}+\sum_{n=1}^{\infty}\left[A_n\tfrac{\cos nx}{\sqrt{\pi}}+B_n\tfrac{\sin nx}{\sqrt{\pi}}\right]$$
$$=\tfrac{1}{\sqrt{2\pi}}\int_a^{a+2\pi}\tfrac{f(x)}{\sqrt{2\pi}}\,\mathrm{d}x+\sum_{n=1}^{\infty}\left[\tfrac{\cos nx}{\sqrt{\pi}}\int_a^{a+2\pi}f(x)\tfrac{\cos nx}{\sqrt{\pi}}\,\mathrm{d}x+\tfrac{\sin nx}{\sqrt{\pi}}\int_a^{a+2\pi}f(x)\tfrac{\sin nx}{\sqrt{\pi}}\,\mathrm{d}x\right]$$
$$=\tfrac{1}{2\pi}\int_a^{a+2\pi}f(x)\,\mathrm{d}x+\sum_{n=1}^{\infty}\left[\tfrac{\cos nx}{\pi}\int_a^{a+2\pi}f(x)\cos nx\,\mathrm{d}x+\tfrac{\sin nx}{\pi}\int_a^{a+2\pi}f(x)\sin nx\,\mathrm{d}x\right]$$
$$=\tfrac{a_0}{2}+\sum_{n=1}^{\infty}[a_n\cos nx+b_n\sin nx],\ \text{for } x\in\langle a\,;\,a+2\pi\rangle.$$

**Theorem.**

The **Fourier series** of a function $f(x) \in R_{\langle a\,;\,a+2l\rangle}$, $a \in R$, $l > 0$ is given by

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right], \quad \text{for } x \in \langle a\,;\, a+2l\rangle,$$

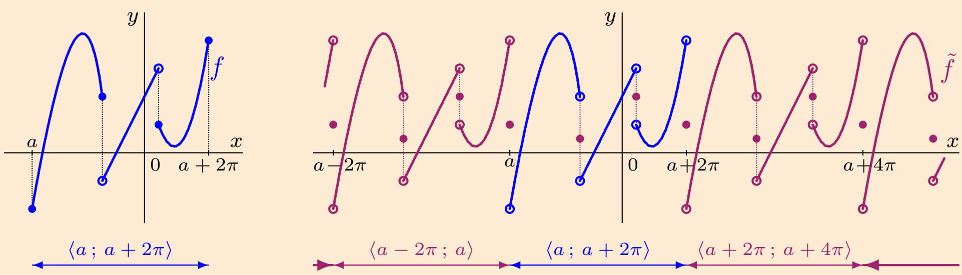
where

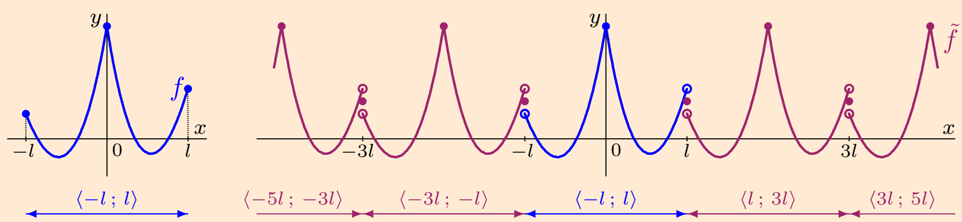
$$a_0 = \frac{1}{l}\int_a^{a+2l} f(x)\,\mathrm{d}x, \;\; a_n = \frac{1}{l}\int_a^{a+2l} f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x, \;\; b_n = \frac{1}{l}\int_a^{a+2l} f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x, \;\; n \in N.$$

A Fourier series converges to the function $\tilde{f}(x)$, $x \in R$:

$$\tilde{f}(x_0) = \begin{cases} \frac{1}{2}\Big[\lim\limits_{x\to x_0^-} f(x) + \lim\limits_{x\to x_0^+} f(x)\Big], & x_0 \in (a\,;\, a+2l), \\ f(x_0 - 2kl), & x_0 \in (a+2kl\,;\, a+2l+2kl),\ k \in Z, \\ \frac{1}{2}\Big[\lim\limits_{x\to a^+} f(x) + \lim\limits_{x\to a+2l^-} f(x)\Big], & x_0 = a+2kl,\ k \in Z. \end{cases}$$

Function $\tilde{f}(x)$ is a periodic function with period $2l$ and equals to the original function at the interval $\langle a\,;\, a+2l\rangle$ at points of continuity or to the average of the two limits at points of discontinuity.




beerb@frcatel.fri.uniza.sk 

**Theorem.**

If $f(x) \in R_{\langle -\pi\,;\,\pi \rangle}$ is an odd function, then $a_n = 0$, $n = 0, 1, 2, \dots$ and the Fourier series collapses to **Fourier Sine Series**

$$f(x) \approx \sum_{n=1}^{\infty} b_n \sin nx, \quad \text{for } x \in \langle -\pi\,;\,\pi \rangle,$$

where

$$b_n = \tfrac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \,\mathrm{d}x = \tfrac{2}{\pi} \int_{0}^{\pi} f(x) \sin nx \,\mathrm{d}x, \ n \in N,$$

respectively

for an odd function $f(x) \in R_{\langle -l\,;\,l \rangle}$, $l > 0$ the Fourier series collapses to **Sine Series**

$$f(x) \approx \sum_{n=1}^{\infty} b_n \sin \tfrac{n\pi x}{l}, \quad \text{for } x \in \langle -l\,;\,l \rangle,$$

where

$$b_n = \tfrac{1}{l} \int_{-l}^{l} f(x) \sin \tfrac{n\pi x}{l} \,\mathrm{d}x = \tfrac{2}{l} \int_{0}^{l} f(x) \sin \tfrac{n\pi x}{l} \,\mathrm{d}x, \ n \in N.$$

**Theorem.**

If $f(x) \in R_{\langle -\pi\,;\,\pi \rangle}$ is an even function, then $b_n = 0$, $n = 1, 2, \dots$ and the Fourier series collapses to **Fourier Cosine Series**

$$f(x) \approx \tfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \quad \text{for } x \in \langle -\pi\,;\,\pi \rangle,$$

where

$$a_0 = \tfrac{1}{\pi} \int_{-\pi}^{\pi} f(x) \,\mathrm{d}x = \tfrac{2}{\pi} \int_{0}^{\pi} f(x) \,\mathrm{d}x,$$

$$a_n = \tfrac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \,\mathrm{d}x = \tfrac{2}{\pi} \int_{0}^{\pi} f(x) \cos nx \,\mathrm{d}x, \ n \in N,$$

respectively

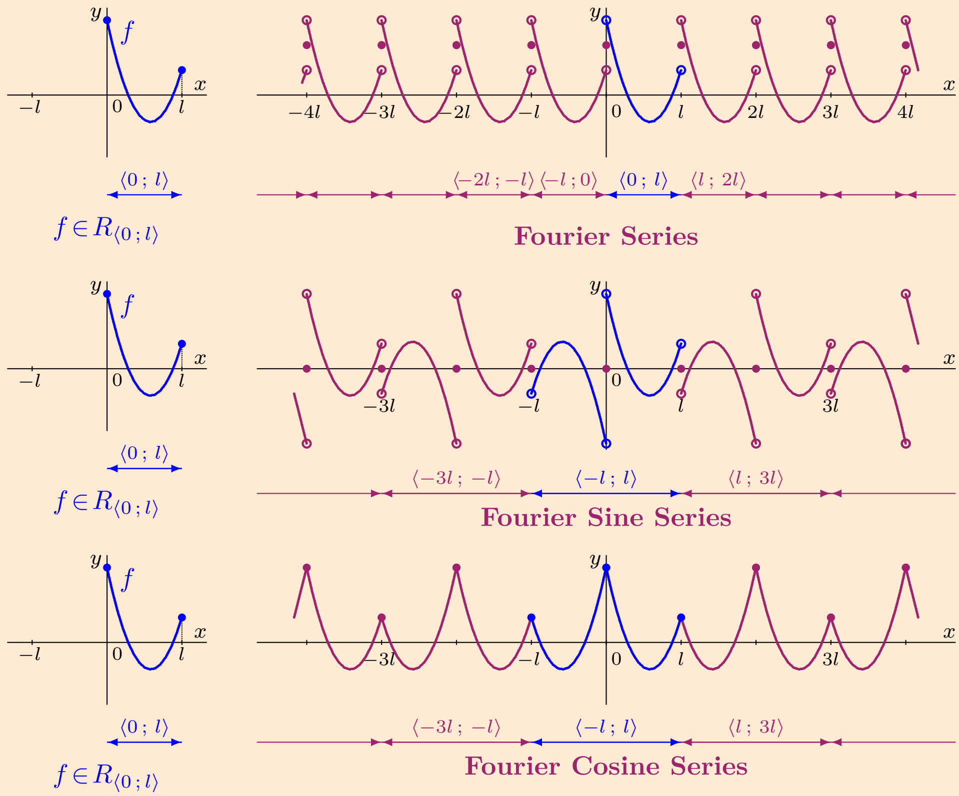
for an even function $f(x) \in R_{\langle -l\,;\,l \rangle}$, $l > 0$ the Fourier series collapses to **Cosine Series**

$$f(x) \approx \tfrac{a_0}{2} + \sum_{n=1}^{\infty} b_n \sin \tfrac{n\pi x}{l}, \quad \text{for } x \in \langle -l\,;\,l \rangle,$$

where

$$a_0 = \tfrac{1}{l} \int_{-l}^{l} f(x) \,\mathrm{d}x = \tfrac{2}{l} \int_{0}^{l} f(x) \,\mathrm{d}x,$$

$$a_n = \tfrac{1}{l} \int_{-l}^{l} f(x) \cos \tfrac{n\pi x}{l} \,\mathrm{d}x = \tfrac{2}{l} \int_{0}^{l} f(x) \cos \tfrac{n\pi x}{l} \,\mathrm{d}x, \ n \in N.$$




## Fourier Series of the function $f(x) = 1$, $x \in \langle 0\,;\,2\pi\rangle$

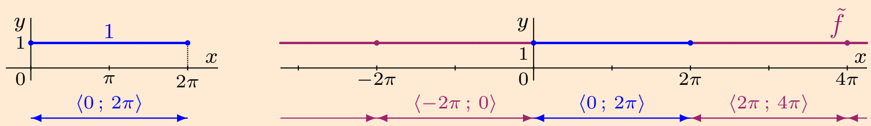
$$f(x) \approx \tilde{f}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \big[a_n \cos nx + b_n \sin nx\big] = \frac{2}{2} + \sum_{n=1}^{\infty} \big[0 \cdot \cos nx + 0 \cdot \sin nx\big] = 1,$$
$x \in \langle 0\,;\,2\pi\rangle$.

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} \mathrm{d}x = \frac{2\pi}{\pi} = 2,$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \cos nx \,\mathrm{d}x = \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_0^{2\pi} = \frac{1}{\pi}\left[\frac{\sin 2n\pi}{n} + \frac{\sin 0}{n}\right] = \frac{1}{\pi}\Big[0 - 0\Big] = 0, \text{ for } n \in N.$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} \sin nx \,\mathrm{d}x = \frac{1}{\pi}\left[-\frac{\cos nx}{n}\right]_0^{2\pi} = \frac{1}{\pi}\left[-\frac{\cos 2n\pi}{n} + \frac{\cos 0}{n}\right] = \frac{1}{\pi}\left[-\frac{1}{n} + \frac{1}{n}\right] = 0,$$
for $n \in N$.



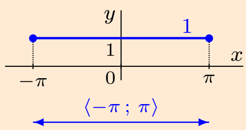
## Fourier (Cosine) Series of the function $f(x)=1,\ x\in\langle-\pi\,;\,\pi\rangle$

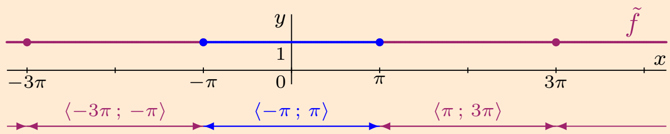
$$f(x)\approx\tilde f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx=\frac{2}{2}+\sum_{n=1}^{\infty}0\cdot\cos nx=1+\sum_{n=1}^{\infty}0=1,\ x\in\langle-\pi\,;\,\pi\rangle.$$

$b_n=0$, for $n\in N$ ($f$ is a even function).

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}\mathrm{d}x=\frac{2}{\pi}\int_{0}^{\pi}\mathrm{d}x=\frac{2\pi}{\pi}=2,$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\cos nx\,\mathrm{d}x\ a_n=\frac{2}{\pi}\int_{0}^{\pi}\cos nx\,\mathrm{d}x=\frac{2}{\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi}=\frac{2}{\pi}\left[\frac{\sin n\pi}{n}+\frac{\sin 0}{n}\right]=\left[\begin{matrix}\sin n\pi=0\\ n\in N\end{matrix}\right]$$
$$=\frac{2}{\pi}\Big[0-0\Big]=0,\ \text{for } n\in N.$$




## Fourier Sine Series of the function $f(x)=1,\ x\in\langle0\,;\,\pi\rangle$

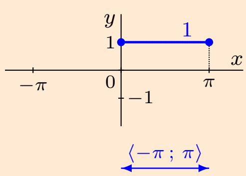
$$f(x)\approx\tilde f_o(x)=\sum_{n=1}^{\infty}b_n\sin nx=\sum_{n=1}^{\infty}\frac{2}{\pi}\frac{(-1)^{n+1}+1}{n}\sin nx=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\big[(-1)^{n+1}+1\big]\sin nx}{n}$$
$$=\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\sin(2k-1)x}{2k-1},\ x\in\langle0\,;\,\pi\rangle.$$

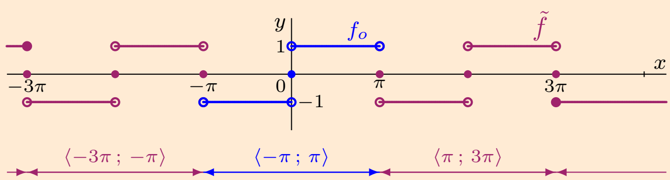
$$f(x)=1,\ x\in\langle0\,;\,\pi\rangle,\quad f_o(x)=\begin{cases}1, & \text{for } x\in(0\,;\,\pi\rangle,\\ 0, & \text{for } x=0,\\ -1, & \text{for } x\in\langle-\pi\,;\,0).\end{cases}$$

$a_0=a_n=0$, for $n\in N$ ($f_o$ is a odd function).

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f_o(x)\sin nx\,\mathrm{d}x=\frac{2}{\pi}\int_{0}^{\pi}f_o(x)\sin nx\,\mathrm{d}x=\frac{2}{\pi}\int_{0}^{\pi}\sin nx\,\mathrm{d}x=\frac{2}{\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi}$$
$$=\frac{2}{\pi}\left[-\frac{\cos n\pi}{n}+\frac{\cos 0}{n}\right]=\left[\begin{matrix}\cos n\pi=(-1)^n\\ n\in N\end{matrix}\right]=\frac{2}{\pi}\left[-\frac{(-1)^n}{n}+\frac{1}{n}\right]=\frac{2}{\pi}\frac{(-1)^{n+1}+1}{n},\ \text{for } n\in N,$$

$$b_n=\begin{cases}\frac{2}{\pi}\frac{(-1)^{2k}+1}{2k-1}=\frac{2}{\pi}\frac{1+1}{2k-1}=\frac{4}{\pi}\frac{1}{2k-1}, & \text{for } n=2k-1,\\ \frac{2}{\pi}\frac{(-1)^{2k+1}+1}{2k}=\frac{2}{\pi}\frac{-1+1}{2k}=0, & \text{for } n=2k,\ k\in N.\end{cases}$$



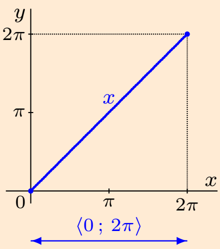
## Fourier Series of the function $f(x)=x,\ x\in\langle 0\,;\,2\pi\rangle$

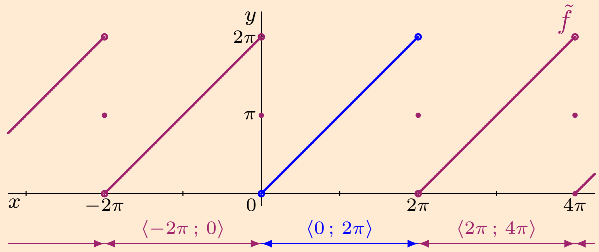
$$f(x)\approx\tilde f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\big[a_n\cos nx+b_n\sin nx\big]=\frac{2\pi}{2}+\sum_{n=1}^{\infty}\Big[0\cdot\cos nx-\frac{2}{n}\sin nx\Big]$$
$$=\pi-2\sum_{n=1}^{\infty}\frac{\sin nx}{n},\ x\in\langle 0\,;\,2\pi\rangle.$$

$$a_0=\frac{1}{\pi}\int_0^{2\pi}x\,\mathrm{d}x=\frac{1}{\pi}\Big[\frac{x^2}{2}\Big]_0^{2\pi}=\frac{1}{\pi}\Big[\frac{4\pi^2}{2}-\frac{0}{2}\Big]=2\pi,$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}x\cos nx\,\mathrm{d}x=\big[\text{ p. 60 }\big]=\frac{1}{\pi}\Big[\frac{\cos nx}{n^2}+\frac{x\sin nx}{n}\Big]_0^{2\pi}$$
$$=\frac{1}{\pi}\Big[\frac{\cos 2n\pi}{n^2}+\frac{2\pi\sin 2n\pi}{n}-\frac{\cos 0}{n^2}-\frac{0\cdot\sin 0}{n}\Big]=\frac{1}{\pi}\Big[\frac{1}{n^2}+\frac{2\pi\cdot 0}{n}-\frac{1}{n^2}-0\Big]=0,\ \text{for } n\in N.$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}x\sin nx\,\mathrm{d}x=\big[\text{ p. 58 }\big]=\frac{1}{\pi}\Big[\frac{\sin nx}{n^2}-\frac{x\cos nx}{n}\Big]_0^{2\pi}$$
$$=\frac{1}{\pi}\Big[\frac{\sin 2n\pi}{n^2}-\frac{2\pi\cos 2n\pi}{n}-\frac{\sin 0}{n^2}+\frac{0\cdot\cos 0}{n}\Big]=\frac{1}{\pi}\Big[\frac{0}{n^2}-\frac{2\pi\cdot 1}{n}-0+0\Big]=-\frac{2}{n},\ \text{for } n\in N.$$



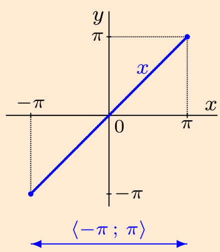



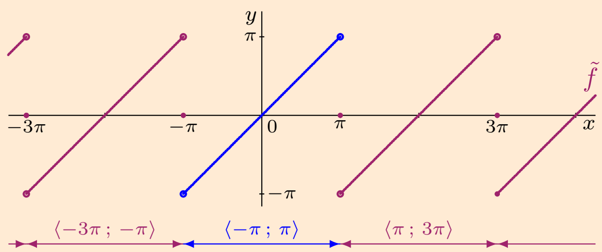

## Fourier (Sine) Series of the function $f(x)=x,\ x\in\langle -\pi\,;\,\pi\rangle$

$$f(x)\approx\tilde f(x)=\sum_{n=1}^{\infty}b_n\sin nx=\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\cdot\sin nx=2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin nx}{n},\ x\in\langle -\pi\,;\,\pi\rangle.$$

$a_0=a_n=0$, for $n\in N$ ($f$ is a odd function).

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}x\sin nx\,\mathrm{d}x=\frac{2}{\pi}\int_0^{\pi}x\sin nx\,\mathrm{d}x=\big[\text{ p. 58 }\big]=\frac{2}{\pi}\Big[\frac{\sin nx}{n^2}-\frac{x\cos nx}{n}\Big]_0^{\pi}$$
$$=\frac{2}{\pi}\Big[\frac{\sin n\pi}{n^2}-\frac{\pi\cos n\pi}{n}-\frac{\sin 0}{n^2}+\frac{0\cdot\cos 0}{n}\Big]=\Big[\begin{matrix}\cos n\pi=(-1)^n\\ \sin n\pi=0,\ n\in N\end{matrix}\Big]=\frac{2}{\pi}\Big[\frac{0}{n^2}-\frac{\pi(-1)^n}{n}-0+0\Big]$$
$$=\frac{2(-1)^{n+1}}{n},\ \text{for } n\in N.$$



## Fourier Cosine Series of the function $f(x)=x$, $x\in\langle 0\,;\,\pi\rangle$

$$f(x)\approx\tilde{f}_e(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx=\frac{\pi}{2}+\sum_{n=1}^{\infty}\frac{2}{\pi}\frac{(-1)^n-1}{n^2}\cos nx$$
$$=\frac{\pi}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{[(-1)^n-1]\cos nx}{n^2}=\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2k-1)x}{(2k-1)^2},\ x\in\langle 0\,;\,\pi\rangle.$$

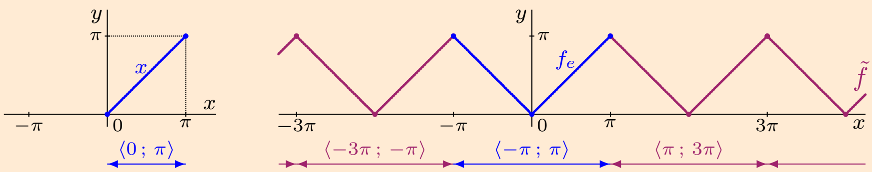
$$f(x)=x,\ x\in\langle 0\,;\,\pi\rangle,\quad f_e(x)=|x|=\begin{cases}x, & \text{for } x\in\langle 0\,;\,\pi\rangle,\\ -x, & \text{for } x\in\langle -\pi\,;\,0\rangle.\end{cases}$$

$b_n=0$, for $n\in N$ ($f_e$ is a even function).

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f_e(x)\,\mathrm{d}x=\frac{2}{\pi}\int_0^{\pi}f_e(x)\,\mathrm{d}x=\frac{2}{\pi}\int_0^{\pi}x\,\mathrm{d}x=\frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}=\frac{2}{\pi}\left[\frac{\pi^2}{2}-\frac{0}{2}\right]=\pi,$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f_e(x)\cos nx\,\mathrm{d}x=\frac{2}{\pi}\int_0^{\pi}f_e(x)\cos nx\,\mathrm{d}x=\frac{2}{\pi}\int_0^{\pi}x\cos nx\,\mathrm{d}x=\left[\text{ p. 60 }\right]$$
$$=\frac{2}{\pi}\left[\frac{\cos nx}{n^2}+\frac{x\sin nx}{n}\right]_0^{\pi}=\frac{2}{\pi}\left[\frac{\cos n\pi}{n^2}+\frac{\pi\sin n\pi}{n}-\frac{\cos 0}{n^2}-\frac{0\cdot\sin 0}{n}\right]=\left[\begin{matrix}\cos n\pi=(-1)^n\\ \sin n\pi=0,\ n\in N\end{matrix}\right]$$
$$=\frac{2}{\pi}\left[\frac{(-1)^n}{n^2}+\frac{\pi\cdot 0}{n}-\frac{1}{n^2}-0\right]=\frac{2}{\pi}\frac{(-1)^n-1}{n^2},\ \text{for } n\in N.$$

$$a_n=\begin{cases}\frac{2}{\pi}\frac{(-1)^{2k-1}-1}{(2k-1)^2}=\frac{2}{\pi}\frac{-1-1}{(2k-1)^2}=-\frac{4}{\pi}\frac{1}{(2k-1)^2}, & \text{for } n=2k-1,\\ \frac{2}{\pi}\frac{(-1)^{2k}-1}{(2k)^2}=\frac{2}{\pi}\frac{1-1}{(2k)^2}=0, & \text{for } n=2k,\ k\in N.\end{cases}$$




## Fourier Series of the function $f(x)=x^2$, $x\in\langle 0\,;\,2\pi\rangle$

$$f(x)\approx\tilde{f}(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos nx+b_n\sin nx\right]=\frac{8\pi^2}{3\cdot 2}+\sum_{n=1}^{\infty}\left[\frac{4}{n^2}\cos nx-\frac{4\pi}{n}\sin nx\right]$$
$$=\frac{4\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{\cos nx-n\pi\sin nx}{n^2},\ x\in\langle 0\,;\,2\pi\rangle.$$

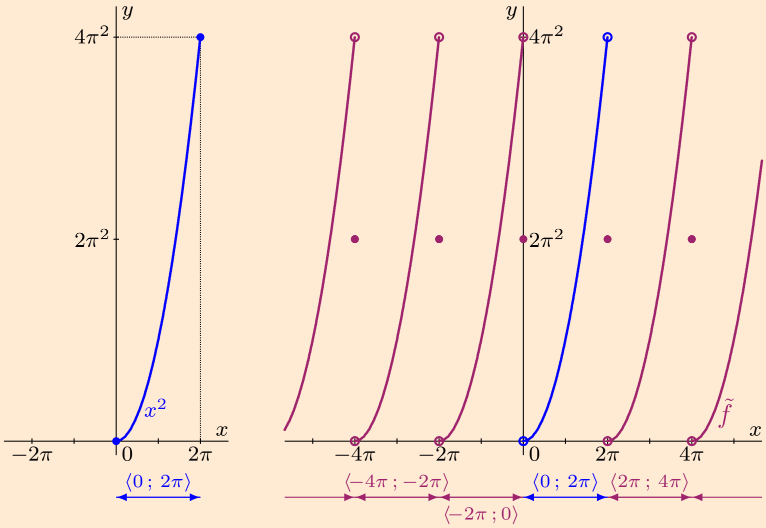
$$a_0=\frac{1}{\pi}\int_0^{2\pi}x^2\,\mathrm{d}x=\frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{2\pi}=\frac{1}{\pi}\left[\frac{8\pi^3}{3}-\frac{0}{3}\right]=\frac{1}{\pi}\frac{8\pi^3}{3}=\frac{8\pi^2}{3},$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}x^2\cos nx\,\mathrm{d}x=\left[\text{ p. 60 }\right]=\frac{1}{\pi}\left[\frac{2x\cos nx}{n^2}-\frac{2\sin nx}{n^3}+\frac{x^2\sin nx}{n}\right]_0^{2\pi}$$
$$=\frac{1}{\pi}\left[\frac{2\cdot 2\pi\cos 2n\pi}{n^2}-\frac{2\sin 2n\pi}{n^3}+\frac{4\pi^2\sin 2n\pi}{n}-\frac{2\cdot 0\cdot\cos 0}{n^2}+\frac{2\sin 0}{n^3}-\frac{0^2\sin 0}{n}\right]$$
$$=\frac{1}{\pi}\left[\frac{2\cdot 2\pi\cdot 1}{n^2}-\frac{2\cdot 0}{n^3}+\frac{4\pi^2\cdot 0}{n}-0+0-0\right]=\frac{1}{\pi}\frac{4\pi}{n^2}=\frac{4}{n^2},\ \text{for } n\in N.$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}x^2\sin nx\,\mathrm{d}x=\left[\text{ p. 59 }\right]=\frac{1}{\pi}\left[\frac{2x\sin nx}{n^2}+\frac{2\cos nx}{n^3}-\frac{x^2\cos nx}{n}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{2\cdot 2\pi \sin 2n\pi}{n^2} + \frac{2\cos 2n\pi}{n^3} - \frac{4\pi^2\cos 2n\pi}{n} - \frac{2\cdot 0\cdot \sin 0}{n^2} - \frac{2\cos 0}{n^3} + \frac{0^2\cos 0}{n}\right]$$

$$= \frac{1}{\pi}\left[\frac{2\cdot 2\pi\cdot 0}{n^2} + \frac{2\cdot 1}{n^3} - \frac{4\pi^2\cdot 1}{n} - 0 - \frac{2\cdot 1}{n^3} + 0\right] = -\frac{1}{\pi}\frac{4\pi^2}{n} = -\frac{4\pi}{n}, \text{ for } n\in N.$$




## Fourier (Cosine) Series of the function $f(x) = x^2$, $x\in\langle -\pi\,;\,\pi\rangle$

$$f(x) \approx \tilde{f}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx = \frac{2\pi^2}{3\cdot 2} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos nx = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n\cos nx}{n^2},$$

$x\in\langle -\pi\,;\,\pi\rangle$.

...........................................................................................

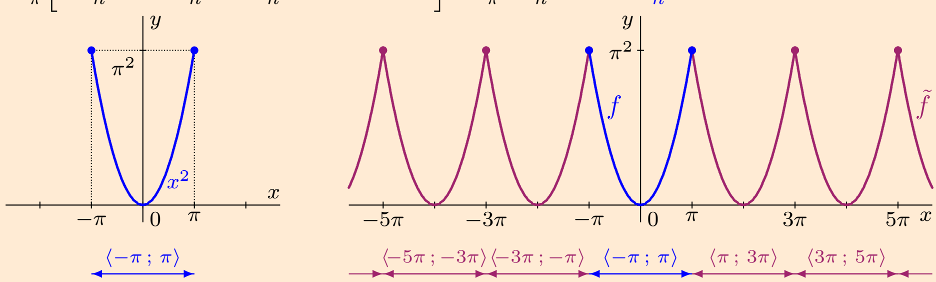
$b_n = 0$, for $n\in N$ ($f$ is a even function).

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} x^2\,\mathrm{d}x = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{\pi^3}{3} - \frac{0}{3}\right] = \frac{2\pi^2}{3},$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,\mathrm{d}x = \left[\text{ p. 60 }\right] = \frac{2}{\pi}\left[\frac{2x\cos nx}{n^2} - \frac{2\sin nx}{n^3} + \frac{x^2\sin nx}{n}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{2\pi\cos n\pi}{n^2} - \frac{2\sin n\pi}{n^3} + \frac{\pi^2\sin n\pi}{n} - \frac{2\cdot 0\cdot\cos 0}{n^2} + \frac{2\sin 0}{n^3} - \frac{0^2\sin 0}{n}\right] = \left[\begin{array}{c}\cos n\pi = (-1)^n\\ \sin n\pi = 0,\ n\in N\end{array}\right]$$

$$= \frac{2}{\pi}\left[\frac{2\pi\cdot(-1)^n}{n^2} - \frac{2\cdot 0}{n^3} + \frac{\pi^2\cdot 0}{n} - 0 + 0 - 0\right] = \frac{2}{\pi}\frac{2\pi(-1)^n}{n^2} = \frac{4(-1)^n}{n^2}, \text{ for } n\in N.$$



## Fourier Sine Series of the function $f(x)=x^2,\ x\in\langle 0\,;\,\pi\rangle$

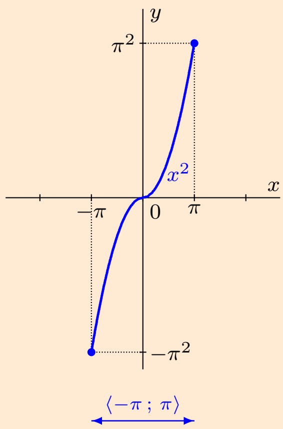
$$f(x)\approx\tilde f_o(x)=\sum_{n=1}^{\infty}b_n\sin nx=\sum_{n=1}^{\infty}\tfrac{2}{\pi}\Big[\tfrac{2(-1)^n-2}{n^3}-\tfrac{\pi^2(-1)^n}{n}\Big]\sin nx$$
$$=\tfrac{2}{\pi}\sum_{n=1}^{\infty}\Big[\tfrac{2(-1)^n-2}{n^3}-\tfrac{\pi^2(-1)^n}{n}\Big]\sin nx,\ x\in\langle 0\,;\,\pi\rangle.$$

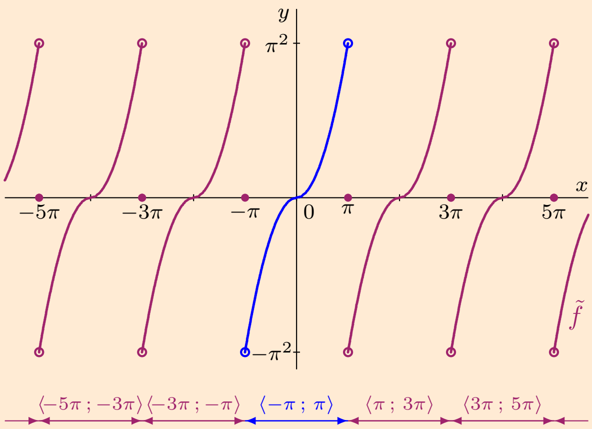
. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

$f(x)=x^2,\ x\in\langle 0\,;\,\pi\rangle,\quad f_o(x)=x\cdot|x|=\begin{cases}x^2, & \text{for } x\in\langle 0\,;\,\pi\rangle\,,\\ -x^2, & \text{for } x\in\langle -\pi\,;\,0\rangle\,.\end{cases}$

$a_0=a_n=0$, for $n\in N$ ($f_o$ is a odd function).

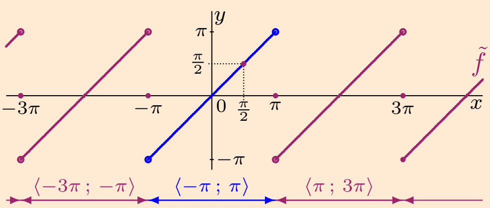
$$b_n=\tfrac{1}{\pi}\int_{-\pi}^{\pi}f_o(x)\cos nx\,\mathrm{d}x=\tfrac{2}{\pi}\int_0^{\pi}f_o(x)\cos nx\,\mathrm{d}x=\tfrac{2}{\pi}\int_0^{\pi}x^2\sin nx\,\mathrm{d}x=\big[\text{ p. 59 }\big]$$
$$=\tfrac{2}{\pi}\Big[\tfrac{2x\sin nx}{n^2}+\tfrac{2\cos nx}{n^3}-\tfrac{x^2\cos nx}{n}\Big]_0^{\pi}$$
$$=\tfrac{2}{\pi}\Big[\tfrac{2\cdot\pi\sin n\pi}{n^2}+\tfrac{2\cos n\pi}{n^3}-\tfrac{\pi^2\cos n\pi}{n}-\tfrac{2\cdot 0\cdot\sin 0}{n^2}-\tfrac{2\cos 0}{n^3}+\tfrac{0^2\cos 0}{n}\Big]=\Big[\begin{matrix}\cos n\pi=(-1)^n\\ \sin n\pi=0,\ n\in N\end{matrix}\Big]$$
$$=\tfrac{2}{\pi}\Big[\tfrac{2\cdot\pi\cdot 0}{n^2}+\tfrac{2(-1)^n}{n^3}-\tfrac{\pi^2(-1)^n}{n}-0-\tfrac{2\cdot 1}{n^3}+0\Big]=\tfrac{2}{\pi}\Big[\tfrac{2(-1)^n-2}{n^3}-\tfrac{\pi^2(-1)^n}{n}\Big],\ \text{for } n\in N.$$




## Some Applications

$f(x)=x,\ x\in\langle-\pi\,;\,\pi\rangle,\ \tilde f(x)=:F(x)=2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin nx}{n},\ x\in R$ (p. 74).



$\boxed{x=\pi}$ $f(\pi)=\pi,\ \tilde{f}(\pi)=\frac{f(-\pi)+f(\pi)}{2}=\frac{-\pi+\pi}{2}=0.$

$$\tilde{f}(\pi)=2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin n\pi}{n}=2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cdot 0}{n}=2\sum_{n=1}^{\infty}0=0.$$

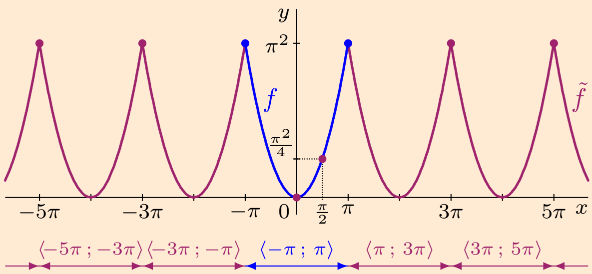
$\boxed{x=\frac{\pi}{2}}$ $f(\frac{\pi}{2})=\tilde{f}(\frac{\pi}{2})=\frac{\pi}{2}.$

$$\frac{\pi}{2}=\tilde{f}(\tfrac{\pi}{2})=2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\frac{n\pi}{2}}{n}=2\sum_{k=1}^{\infty}\Big[\frac{(-1)^{2k-1+1}\sin\frac{(2k-1)\pi}{2}}{2k-1}+\frac{(-1)^{2k+1}\sin\frac{2k\pi}{2}}{2k}\Big]$$
$$=2\sum_{k=1}^{\infty}\Big[\frac{\sin\frac{(2k-1)\pi}{2}}{2k-1}-\frac{\sin k\pi}{2k}\Big]=\left[\begin{array}{c}\sin\frac{(2k-1)\pi}{2}=(-1)^{k+1}\\ \sin k\pi=0,\ k\in N\end{array}\right]=2\sum_{k=1}^{\infty}\Big[\frac{(-1)^{k+1}}{2k-1}-0\Big]=2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2k-1}.$$

$$\frac{\pi}{2}=\tilde{f}(\tfrac{\pi}{2})=2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\frac{n\pi}{2}}{n}$$
$$=2\Big[\sin\tfrac{\pi}{2}-\tfrac{1}{2}\sin\tfrac{2\pi}{2}+\tfrac{1}{3}\sin\tfrac{3\pi}{2}-\tfrac{1}{4}\sin\tfrac{4\pi}{2}+\tfrac{1}{5}\sin\tfrac{5\pi}{2}-\tfrac{1}{6}\sin\tfrac{6\pi}{2}+\tfrac{1}{7}\sin\tfrac{7\pi}{2}-\tfrac{1}{8}\sin\tfrac{8\pi}{2}+\cdots\Big]$$
$$=2\Big[1-\tfrac{1}{2}\cdot 0+\tfrac{1}{3}\cdot(-1)-\tfrac{1}{4}\cdot 0+\tfrac{1}{5}\cdot 1-\tfrac{1}{6}\cdot 0+\tfrac{1}{7}\cdot(-1)-\tfrac{1}{8}\cdot 0+\cdots\Big]$$
$$=2\Big[1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\cdots\Big]=2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2k-1}\quad\Rightarrow\quad\frac{\pi}{4}=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2k-1}.$$

---

$g(x)=x^2,\ x\in\langle-\pi\,;\,\pi\rangle,\ \tilde{g}(x)=:G(x)=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n\cos nx}{n^2},\ x\in R$ (p. 76).




---

$\boxed{x=0}$ $g(0)=\tilde{g}(0)=0.$

$$0=\tilde{g}(0)=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n\cos(n\cdot 0)}{n^2}=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n\cdot 1}{n^2}=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$
$$=\frac{\pi^2}{3}-4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\quad\Rightarrow\quad\frac{\pi^2}{12}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}.$$

$\boxed{x=\frac{\pi}{2}}$ $g(\frac{\pi}{2})=\tilde{g}(\frac{\pi}{2})=\frac{\pi^2}{4}.$

$$\frac{\pi^2}{4}=\tilde{g}(\tfrac{\pi}{2})=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n\cos\frac{n\pi}{2}}{n^2}=\frac{\pi^2}{3}+4\sum_{k=1}^{\infty}\Big[\frac{(-1)^{2k-1}\cos\frac{(2k-1)\pi}{2}}{(2k-1)^2}+\frac{(-1)^{2k}\cos\frac{2k\pi}{2}}{(2k)^2}\Big]$$
$$=\frac{\pi^2}{3}+4\sum_{k=1}^{\infty}\Big[\frac{\cos k\pi}{4k^2}-\frac{\cos\frac{(2k-1)\pi}{2}}{(2k-1)^2}\Big]=\left[\begin{array}{c}\cos\frac{(2k-1)\pi}{2}=0\\ \cos k\pi=(-1)^k,\ k\in N\end{array}\right]=\frac{\pi^2}{3}+4\sum_{k=1}^{\infty}\Big[\frac{(-1)^k}{4k^2}-0\Big]$$
$$=\frac{\pi^2}{3}-\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2}\quad\Rightarrow\quad\frac{\pi^2}{4}-\frac{\pi^2}{3}=-\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2}\quad\Rightarrow\quad\frac{\pi^2}{12}=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2}.$$

$\boxed{x=\pi}$ $g(\pi)=\tilde{g}(\pi)=\pi^2.$

$$\pi^2=\tilde{g}(\pi)=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n\cos n\pi}{n^2}=\left[\begin{matrix}\cos n\pi=(-1)^n\\ n\in N\end{matrix}\right]=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n\cdot(-1)^n}{n^2}$$
$$=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{1}{n^2}\qquad\Rightarrow\quad\frac{2\pi^2}{3}=4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\qquad\Rightarrow\quad\frac{\pi^2}{6}=\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}=\frac{\pi}{4},\qquad\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=\frac{\pi^2}{12},\qquad\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

---

$(x^2)'=2x$, for $x\in R$, i. e. $g'(x)=2f(x)$, for $x\in\langle-\pi\,;\,\pi\rangle$.

$$G'(x)=\left[\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n\cos nx}{n^2}\right]'=0+4\left[\sum_{n=1}^{\infty}\frac{(-1)^n\cos nx}{n^2}\right]'=4\sum_{n=1}^{\infty}\left[\frac{(-1)^n\cos nx}{n^2}\right]'$$
$$=4\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n^2}(-\sin nx)\cdot n\right]=4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin nx}{n}=2\cdot 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin nx}{n}=2F(x),$$
for $x\in R$.

---

$x^2=\int_0^x 2t\,\mathrm{d}t=2\int_0^x t\,\mathrm{d}t$, for $x\in R$, i. e. $g(x)=2\int_0^x f(t)\,\mathrm{d}t$, for $x\in\langle-\pi\,;\,\pi\rangle$.

$$2\int_0^x F(t)\,\mathrm{d}t=2\int_0^x\left[2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin nt}{n}\right]\mathrm{d}t=4\int_0^x\left[\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin nt}{n}\right]\mathrm{d}t$$
$$=4\sum_{n=1}^{\infty}\left[\int_0^x\frac{(-1)^{n+1}\sin nt}{n}\,\mathrm{d}t\right]=4\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}}{n}\int_0^x\sin nt\,\mathrm{d}t\right]$$
$$=4\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}}{n}\left[\frac{-\cos nt}{n}\right]_0^x\right]=4\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}}{n}\left[-\frac{\cos nx}{n}+\frac{\cos 0}{n}\right]\right]$$
$$=4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(1-\cos nx)}{n^2}=4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}+4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(-\cos nx)}{n^2}$$
$$=4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}+4\sum_{n=1}^{\infty}\frac{(-1)^n\cos nx}{n^2}=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n\cos nx}{n^2}=G(x),$$
where $4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=\frac{a_0}{2}$, $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}x^2\,\mathrm{d}x=$ [ p. 76 ] $=\frac{2\pi^2}{3}$, for $x\in R$.

---

**Finita la commedia**