Mathematical Analysis supported by wxMaxima

Summer School Open Source

Innovative Open Source courses for Computer Science

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Contents

Introduction to wxMaxima

wxMaxima is a document based interface for the computer algebra system Maxima. wxMaxima provides menus and dialogs for many common maxima commands, autocompletion, inline plots and simple animations. wxMaxima is distributed under the GPL license.

Maxima is one of the Open Source programs with open source code. The program can be compiled in various OS, including Windows, GNU/Linux and MacOS X. A precompiled program for GNU/Linux and Windows is available free of charge on the SourceForge website `https://sourceforge.net/projects/maxima/files/`.

After starting the wxMaxima environment, a menu window will appear on the screen at the top. Below the menu is a space where we can enter commands and where outputs appear.

```
(%i1) First input line.
(%o1) First output line.
(%i2) Second input line.
(%o2) Second output line.
```

We enter commands on separate lines (input lines), their execution is ensured by simultaneously pressing the **Shift** keys and**Enter** or by clicking on in the menu icon (Send the current cell to maxima). Input lines are listed with (%i1) and output lines are listed with (%o1). The numbers for the input line and the corresponding output line are identical and based on this number, we can refer to the content of these lines.

```
(%i1) solve(0=x+2,x);
(%o1) [x = -2]
(%i2) %i1;
(%o2) solve(0 = x + 2, x)
(%i3) %o1;
(%o3) [x = -2]
```

The commands are executed on new separate lines (output lines). Commands on input lines can be terminated with the symbol `;` (which the system will automatically complete) or the `$` symbol, which suppresses the display of the corresponding output.

We can enter more commands on the input line, but we must separate them symbols `;` or `$`. We can also structure the command on multiple input lines.

```
(%i1) a:2;b:3;solve(a*x+b*x^2=0,x)
(a)   2
(b)   3
(%o1) [x = -2/3, x = 0]
(%i2) a:2$ b:3$ solve(a*x+b*x^2=0,x);
(%o2) [x = -2/3, x = 0]
(%i3) a:2$
      b:3$
      solve(a*x+b*x^2=0,x);
```

(%o3) $[x=-\frac{2}{3}, x=0]$

We can save the output in various shapes and then use it in other programs (LaTeX, MSWord equation editor, ...). Output (%o3) from the previous window we can:

- copy (`Crl C` and `Crl V`), resp. copy as text (can be used eg. for MSWord equation editor): `x=-2/3, x=0`,
- copy as LaTeX `\[x=-\frac{2}{3}\operatorname{,}x=0\]`,
- copy as MathML, image, RTF, SVG...

The wxMaxima environment has a well-designed user help, which can be found in the `Help` menu. You can also open Help by pressing the `F1` key. You can also find the manual on the website `https://maxima.sourceforge.io/docs/manual/maxima_369.html`.

Basic Commands

The command `apropos` we can find out the exact name of the command using part of its name.

```
(%i1) apropos("plot")
(%o1) [barsplot,boxplot,contour_plot,get_plot_option,gnuplot,...
```

Command `describe` prints a description of the entered command.

```
(%i1) describe(plot2d);
      -- Function: plot2d
        plot2d (<expr><, <range_x><, <options><)
        plot2d (<expr_<>=<expr_<>, <range_x><, <range_y><, <options><)
        ...
(%o1) true
```

Expressions are entered using the usual characters of operations, sessions, and functions. Arguments of functions and commands are in parentheses, multiplication symbol `*` must be entered! The exponentiation is specified by the character `^` or the pair `**`.

Symbol `:` is used to assign a value to the right of the expression to the left.

```
(%i1) a:2$ b:3$ solve(a*x+b*x^2=0,x);
(%o1) [x=-2/3, x=0]
```

In the menu `View` and submenu `Display equations` we can change display output lines for shapes `in 2D`, `as 1D ASCII` or `as ASCII Art`.

The default display is `in 2D`. You can also change the output settings with the command `set_display`. Setting to shape `in 2D` has argument `none`.

```
(%i1) x/sqrt(x^2+1);set_display('none)$
(%o1) x/sqrt(x^2+1)          /* in 2D */
```

Using the `ascii` argument command `set_display` change the display output to the form `as 1D ASCII` and using the `xml` argument to form `as ASCII Art`.

```
(%i1) x/sqrt(x^2+1);set_display('ascii)$
(%o1) x/sqrt(x^2 + 1)        /* as 1D ASCII */
(%i2) x/sqrt(x^2+1);set_display('xml)$
          x
(%o2) ------------          /* as ASCII Art */
            2
      sqrt(x  + 1)
```

The command `kill` we can remove variables with all their assignments and memory properties.

```
(%i1) kill(a,b)         /* removes all bindings from the arguments a,b */
(%i2) kill(all)         /* removes all items on all infolists */
```

Working with Numbers and Basic Constants

Maxima can work with real numbers written in numerical or symbolic form. The way of writing real numbers can be set in the menu `Numeric` using the switch `Numeric Output` between numeric and symbolic display. Here we can also choose the method and accuracy of numerical display. The setting of the variable `numer` determines the method of displaying.

By default, 16 digits (including the decimal point) are displayed. The display accuracy is defined by the variable `fpproc` and affects the display using `bfloat`. Output `float` always displays the same. We can increase or decrease the accuracy practically indefinitely. We can change it globally and locally for only one variable or command.

```
(%i1) log(2);
(%o1) log(2)
(%i2) log(2),numer;
(%o2) 0.6931471805599453
(%i3) float(log(2));
(%o3) 0.6931471805599453
(%i4) bfloat(log(2));
(%o4) 6.931471805599453b-1
(%i5) log(2),bfloat;
(%o5) 6.931471805599453b-1
(%i6) bfloat(log(2)),fpprec=34;
```

```
(%o6) 6.931471805599453094172321214581766b-1
(%i6) bfloat(log(2)),fpprec=134;
(%o6) 6.93147180559945309417232121458[78digits]102057068573368552023575813b-1
```

Numeric constants e, π, i (imaginary unit) have the prefix `%`, i.e. `%e`, `%pi`, `%i`. This also applies to constants that are part of or the result of calculations. They also have the prefix `%`.

Maxima has predefined constants `inf`, `minf` for real infinite ∞, $-\infty$ and `infinity` for complex infinity.

Logical constants `true` and `false` they represent truth and untruth.

```
(%i1) %pi; %i; %e;
(%o1) π %i %e
(%i2) minf; inf;
(%o2) -∞ ∞
(%i3) infinity;
(%o3) infinity
```

We do not deal with complex numbers in this course, so we will only mention how they are displayed. By default, complex numbers are entered in algebraic form (`rectform`). They can be converted to trigonometric (exponential) form using the command `polarform`.

```
(%i1) z:1+%i;
(z)   i+1
(%i2) polarform(z)+rectform(z);
(%o2) √2 e^(iπ/4) +i+1
```

Assignments and Functions

The `:` operator we use to assign values or expressions to variables. We define functions using the assignment `:=`.

```
(%i1) f(x):=x^2+2*x+3;
(%o1) f(x) := x^2 + 2 * x + 3
(%i6) f(x);f(y);f(x+1);f(-2);f(1);
(%o2) x^2 + 2 * x + 3
(%o3) y^2 + 2 * y + 3
(%o4) (x + 1)^2 + 2 * (x + 1) + 3
(%o5) 3
(%o6) 6
```

Maxima contains many more functions than standard programming languages. These are not only the real functions themselves, but also various functions for their support. The basic functions include `sign(x)`, `abs(x)`, `floor(x)` (bottom whole of x) `round(x)` (rounded

x to the nearest whole number), `truncate(x)` (removes all digits after the decimal point), `ceiling(x)` (upper integer x).

```
(%i2)  f(x):=sign(x)$ print(f(-3.2),f(0),f(3.2))$
       neg zero pos
(%i4)  f(x):=abs(x)$ print(f(-3.2),f(0),f(3.2))$
       3.2  0  3.2
(%i6)  f(x):=floor(x)$ print(f(-3.6),f(-3.2),f(-1),f(0),f(1),f(3.2),f(3.6))$
       −4  −4  −1  0  1  3  3
(%i8)  f(x):=round(x)$ print(f(-3.6),f(-3.2),f(-1),f(0),f(1),f(3.2),f(3.6))$
       −4  −3  −1  0  1  3  4
(%i10) f(x):=truncate(x)$ print(f(-3.6),f(-3.2),f(-1),f(0),f(1),f(3.2),f(3.6))$
       −3  −3  −1  0  1  3  3
(%i12) f(x):=ceiling(x)$ print(f(-3.6),f(-3.2),f(-1),f(0),f(1),f(3.2),f(3.6))$
       −3  −3  −1  0  1  4  4
```

We used the command `print` to format the report.

```
(%i3)  a:2$ b:log(2),numer$ print("Logarithm of a number",a," is ",log(a),"=",b)$
       Logarithm of a number 2 is log(2) = 0.6931471805599453
```

Maxima contains many elementary functions. They are, for example `exp(x)=%e^x`, `log(x)`, trigonometric functions and their inverse functions `sin(x)`, `cos(x)`, `tan(x)`, `cot(x)`, `asin(x)`, `acos(x)`, `atan(x)`, `acot(x)`, hyperbolic functions and their inverse functions `sinh(x)`, `cosh(x)`, `tanh(x)`, `coth(x)` `asinh(x)`, `acosh(x)`, `atanh(x)`, `acoth(x)` etc.

Maxima also includes many features to support them. Some of them are not implemented directly in the wxMaxima environment, but in external libraries called packages. These packages are loaded into the system using the `load` command. We will show the `spangl` package for an example to support work with trigonometric functions.

(%i2) `print(tan(%pi/8),ratsimp(tan(%pi/8)),trigsimp(tan(%pi/8)))$`

$\tan\left(\frac{\pi}{8}\right)\ \tan\left(\frac{\pi}{8}\right)\ \frac{\sin\left(\frac{\pi}{8}\right)}{\cos\left(\frac{\pi}{8}\right)}$

(%i3) `load(spangl);`

(%o3) *../share/trigonometry/spangl.mac*

(%i4) `tan(%pi/8);`

(%o4) $\sqrt{2}-1$

Working with Expressions

Maxima operations and calculations take place in an environment, in which the system presupposes the validity of certain conditions. We may change these terms. Many times we need to change the conditions only locally for a particular calculation without to change global settings. For this purpose, Maxima has a very efficient `ev` command that allows define a specific environment within a single command.

After entering the command `ev(a,b1,b2,..., bn)` the expression `a` is evaluated if the conditions `b1`, `b2`, ..., `bn` are met. These conditions can be equations, assignments, functions, switches (logical settings). The example shows an example of solving a quadratic equation using the command `solve`. Variables `a`, `b`, `c` after executing the command `ev` they do not have values assigned.

(%i1) `ev(solve(a*x^2+b*x+c=0,x),a:2,b:-1,c=-3);`

(%o1) $[x = \frac{3}{2}, x = -1]$

(%i2) `solve(a*x^2+b*x+c=0,x);`

(%o2) $\left[x = -\frac{\sqrt{b^2-4ac}+b}{2a},\ x = \frac{\sqrt{b^2-4ac}-b}{2a}\right]$

Maxima offers several commands for simplifying and editing various expressions. The basic functions can be found in the `Simplify` menu. With the `ratsimp` commands and `trigsimp` we have already met and when adjusting the value of `tan(%pi/8)` they did not have the desired effect.

Maxima offers using the `example` command examples of individual commands. Let's take a look at some of the examples offered by `example(ratsimp)`.

(%i2) `f(x):=b*(a/b-x)+b*x+a$ print(f(x),"?",ratsimp(f(x)))$`

$bx + b(\frac{a}{b} - x) + a\ \ ?\ \ 2a$

(%i3) `ratsimp(a+1/a);`

(%o3) $\frac{a^2+1}{a}$

(%i4) `ev(x^(a+1/a),ratsimp);`

(%o4) $x^{a+\frac{1}{a}}$

(%i5) `ev(x^(a+1/a),ratsimpexpons);`

(%o5) $x^{\frac{a^2+1}{a}}$

Function `expand` multiplies the relevant members in the expression. Function `factor` on the contrary, it decomposes the expression. Function `gfactor` it does so over a field of complex numbers.

(%i1) `f(x):=(x+1)*(x^2-4)*(x^2+4)$`
(%i3) `ratsimp(f(x));expand(f(x));`
(%o2) $x^5 + x^4 - 16x - 16$
(%o3) $x^5 + x^4 - 16x - 16$
(%i6) `factor(f(x));gfactor(f(x));factor(100);`
(%o4) $(x-2)(x+1)(x+2)(x^2+4)$
(%o5) $(x-2)(x+1)(x+2)(x-2\%\mathrm{i})(x+2\%\mathrm{i})$
(%o6) $2^2 5^2$

We decompose a rational polynomial function into partial fractions using the command `partfrac`.

(%i1) `partfrac((x+1)/(x^2-2*x+1),x);`

(%o1) $\frac{1}{x-1} + \frac{2}{(x-1)^2}$

We can substitute expressions using the commands `subst(a,b,c)` and `ratsubst(a,b,c)`. The expression `a` will be replaced by `b` and subsequently substituted into the expression `c`. When using the `subst` command must be `b` the simplest part (atom) or a complete subexpression of the expression `c`. In the example, the subexpression is not `x+y` complete (missing `z`). the `ratsubst` command it also modifies the resulting expression.

(%i2) `subst(x+y,a,a^2+b^2);ratsubst(x+y,a,a^2+b^2);`
(%o1) $(y+x)^2 + b^2$
(%o2) $y^2 + 2xy + x^2 + b^2$
(%i4) `subst(a,x+y,x+y+z);ratsubst(a,x+y,x+y+z);`
(%o3) $z + y + x$
(%o4) $z + a$

Limits and Derivatives

In the `Calulus` menu we find functions for solving basic problems of mathematical analysis (limits, derivation, integration, sums of series, decomposition of a function into a Taylor polynomial. . .).

We calculate the limits using the command`limit`. The last parameter determines the direction of unilateral limits, has the values `plus` or `minus` and is optional. If not specified, Maxima calculates the limit as complex. The commands `limit(f(x),x,a)`, `limit(f(x),x,a,plus)` we calculate the limits $\lim_{x\to a} f(x)$, $\lim_{x\to a^+} f(x)$.

(%i4) `limit(1/x,x,0);limit(1/x,x,0,plus);limit(1/x,x,0,minus);limit(1/x,t,0);`
(%o1) $infinity$
(%o2) ∞
(%o3) $-\infty$
(%o4) $\frac{1}{x}$

If we use apostrophe `'` before the command, the command will not be executed, it will only be displayed.

(%i2) `limit(((1-n)/(1+3*n))^(1+4*n),n,inf);'limit(((1-n)/(1+3*n))^(1+4*n),n,inf);`

(%o1) 0

(%o2) $\lim_{n\to\infty} \left(\frac{1-n}{3n+1}\right)^{4n+1}$

Derivatives are calculated using the command `diff`. The parameter that determines the order of derivation is optional.

```
(%i4) f(x):=2*x^4-3*x+sin(x);
      print("f'=",diff(f(x),x),"=",diff(f(x),x,1))$
      print("f''=",diff(diff(f(x),x),x),"=",diff(f(x),x,2),"=",diff(f(x),x,1,x,1))$
      print("f^(10)=",diff(f(x),x,10),"=",diff(f(x),x,1,x,9))$
```

(%o1) $f(x) := 2x^4 - 3x + \sin(x)$

$f' = \cos(x) + 8x^3 - 3 = \cos(x) + 8x^3 - 3$

$f'' = 24x^2 - \sin(x) = 24x^2 - \sin(x) = 24x^2 - \sin(x)$

$f\_(10) = -\sin(x) = -\sin(x)$

We calculate partial derivatives using the same command.

```
(%i3) g(x,y):=x^3*y^2-1;
      print("g'_x=",diff(g(x,y),x),", resp. g'_y=",diff(g(x,y),y,1))$
      print("g''_(xx)=",diff(g(x,y),x,2),", resp. g''_(yx)=",diff(g(x,y),y,1,x,1))$
(%o1) g(x,y):=x^3*y^2-1
```

$g'\_x = 3x^2y^2, resp.\ g'\_y = 2x^3y$

$g''\_(xx) = 6xy^2, resp.\ g''\_(yx) = 6x^2y$

We calculate the Taylor polynomial nth degree using the command `taylor`. You can find this command in the `Calculus` menu and the `Get Series...` submenu. We calculate Taylor series of functions f degree n in the middle c with the command`taylor(f(x),x,c,n)`. Its coefficients are obtained using the command `coeff`. The use of this command depends on the `taylor` command.

```
(%i1) t1:taylor(sin(x),x,0,5); t2:taylor(sin(x),x,-1,5);
```

(t1) $x - \frac{x^3}{6} + \frac{x^5}{120} + \cdots$

(t2) $-\sin(1) + \cos(1)(x+1) + \frac{\sin(1)(x+1)^2}{2} - \frac{\cos(1)(x+1)^3}{6} - \frac{\sin(1)(x+1)^4}{24} + \frac{\cos(1)(x+1)^5}{120} + \cdots$

```
(%i3) print(coeff(sin(x),x,5)," and ",coeff(t1,x,5)," and ",coeff(t2,x,5))$
```

$0\ and\ \frac{1}{120}\ and\ \frac{\cos(1)}{120}$

Taylor polynomial polynomial is again a polynomial, only it is expressed in another form. Practically, only the coordinate system in which we express the polynomial changes. The beginning of the system moves from point 0 to point -1. In the following example, the Taylor polynomial of a given polynomial is calculated in another way. Command `taylor` gives three points at the end, even if development is closed.

```
(%i1) f(x):=2*x^5-x^4-3*x^3-x+1;
```

(%o1) $f(x) := 2x^5 - x^4 + (-3)x^3 - x + 1$

```
(%i2) tp1:taylor(f(x),x,-1,5);
```

(tp1) $2 + 4(x+1) - 17(x+1)^2 + 21(x+1)^3 - 11(x+1)^4 + 2(x+1)^5 + \cdots$

```
(%i4) ratsimp(tp1);expand(tp1);
```

(%o3) $2x^5 - x^4 - 3x^3 - x + 1$

(%o4) $2x^5 - x^4 - 3x^3 - x + 1$

(%i6)
```
tpx:ratsubst(t,x+1,f(x));subst(x+1,t,tpx);
```

(tpx) $2t^5 - 11t^4 + 21t^3 - 17t^2 + 4t + 2$

(tp2) $2(x+1)^5 - 11(x+1)^4 + 21(x+1)^3 - 17(x+1)^2 + 4(x+1) + 2$

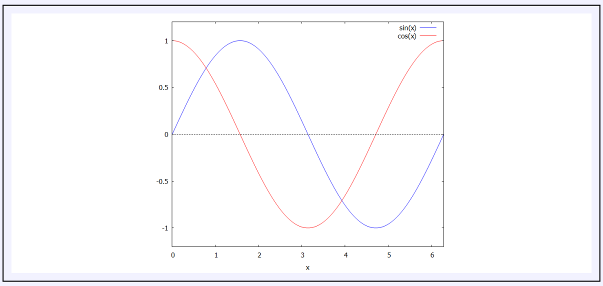
(%i7)
```
tp1-tp2;
```

(%o7) $0 + \cdots$

Function Graphs

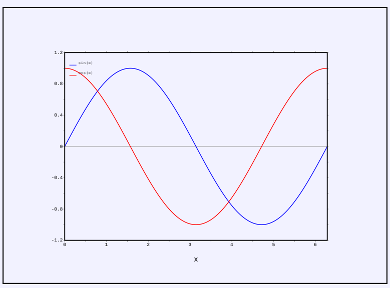
We can plot the function graph in several ways. The easiest way is to choose `Plot` in the menu submenu `Plot 2d ...`. If we choose `Format=gnuplot`, the function is rendered by the command `plot2d` using the Open Source program `Gnuplot` to a new window. `Gnuplot` is automatically installed together with Maxima.

(%i1)
```
plot2d([sin(x),cos(x)],[x,-%pi,2*%pi],[y,-1.2,1.2],[plot_format, gnuplot])$
```



If we choose `Format=wxmaxima`, Maxima will plot the graph using the command `plot2d` to a new window. We can only save the image in script.

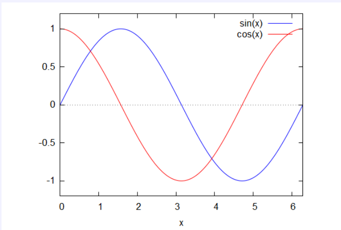
(%i1)
```
plot2d([sin(x),cos(x)],[x,-%pi,2*%pi],[y,-1.2,1.2],[plot_format, xmaxima])$
```



If we choose `Format=inline`, Maxima draws a graph using the command `wxplot2d` into your environment.

(%i1)
```
wxplot2d([sin(x),cos(x)],[x,-%pi,2*%pi],[y,-1.2,1.2])$
```

(%o1)

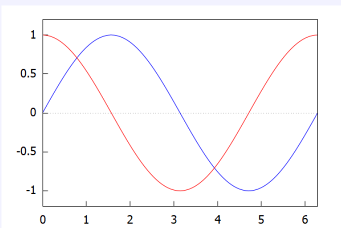


Commands `plot2d` and `wxplot2d` they have the same syntax and have many more parameters. Parameters can be found, for example, with the command `describe(plot2d)`.

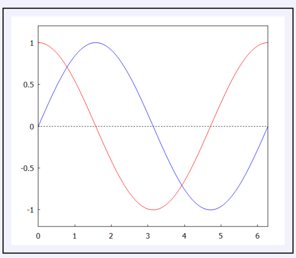
It is better to use the `wxdraw2d` command to print function graphs or `draw2d`, which should be routed to the output of `Gnuplot`. These commands have a slightly different syntax than the `wxplot2d`, `plot2d`. The print parameters are simpler and clearer. The plotted function must be located in the command `explicit`, `parametric` or `implicit`.

(%i1)
```
wxdraw2d(xaxis=true,yaxis=true,xrange=[0,2*%pi],yrange=[-1.2,1.2],
color=blue,explicit((sin(x)),x,0,2*%pi),
color=red,explicit((cos(x)),x,0,2*%pi))$
```

(%o1)

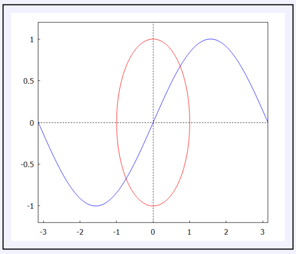


(%i1)
```
draw2d(xaxis=true,yaxis=true,xrange=[0,2*%pi],yrange=[-1.2,1.2],
color=blue,explicit((sin(x)),x,0,2*%pi),
color=red,explicit((cos(x)),x,0,2*%pi))$
```



We plot a parametric curve or function in a similar way.

```
(%i1) draw2d(xaxis=true,yaxis=true,xrange=[-%pi,%pi],yrange=[-1.2,1.2],
      color=blue,explicit((sin(x)),x,-%pi,%pi),
      color=red,nticks=300,parametric(cos(t),sin(t),t,0,2*%pi))$
```



Sequences and Series

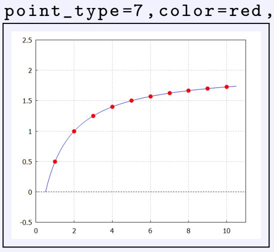
Sequences can be created in Maxima, for example, using the command `makelist` or with the statements of the cycle `for - to`.

Command `makelist` creates a list that we can display as a whole and by members.

```
(%i2) S1:makelist(2*n^2-1,n,1,10);S2:makelist(2*n^2-1,n,2,10,2);
(S1)  [1, 7, 17, 31, 49, 71, 97, 127, 161, 199]
(S2)  [7, 31, 71, 127, 199]
(%i4) S1[1];S1[10];
(%o3) 1
(%o4) 199
```

Arranged pairs are enclosed in square brackets and can be displayed as points in a plane. In the following example, a sequence is also generated with its patterns and then plotted with a command `draw2d`.

```
(%i1) S1:makelist([n,(2*n-1)/(n+1)],n,1,10);
(S1)  [[1, 1/2], [2, 1], [3, 5/4], [4, 7/5], [5, 3/2], [6, 11/7], [7, 13/8], [8, 5/3], [9, 17/10], [10, 19/11]]
(%i2) draw2d(grid=true,xaxis=true,yaxis=true,xrange=[0,11],yrange=[-0.5,2.5],
      color=blue,explicit((2*n-1)/(n+1),n,0.5,10.5),
      point_type=7,color=red,points(S1))$
```



Using the commands`for - to` we will list several members of the sequence $\{2n^2 - 1\}_{n=1}^{\infty}$.

```
(%i1) (for n:1 thru 12 do (a_n: 2*n^2-1, print(a_n)) )$
      1
      7
      17
      31
      49
      71
      97
      127
      161
      199
      241
      287
```

A nice example of using the commands `for - do` is a Fibonacci sequence.

```
(%i3) a0:0$ a1:1$ (for i:1 thru 12 do (an:a1+a0, print(an), a1:a0, a0:an))$
      1
      1
      2
      3
      5
      8
      13
      21
      34
      55
      89
      144
```

We calculate the finite and infinite sum using the command `sum`.

```
(%i1) sum(2*n^2-1,n,1,8);
(%o1) 400
```

With this command, Maxima can calculate the exact sum of some infinite series. The sum of the series can be entered in the menu `Calculus` and the `Calculate Sum...` submenu.

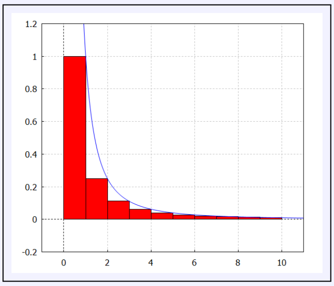
```
(%i2) sum(1/k^2,k,1,inf),simpsum; sum(1/k^2,k,1,inf);
```

(%o1) $\frac{\pi^2}{6}$

(%o2) $\sum_{k=1}^{\infty}\left(\frac{1}{k^2}\right)$

The number series from the previous example can be graphically represented as follows.

```
(%i1) a(n):=1/n^2$ rec:makelist(rectangle([i-1,0],[i,a(i)]),i,1,10)$
      draw2d(grid=true,xaxis=true,yaxis=true,xrange=[-1,11],yrange=[-0.2,1.2],
      border=true,color=black,fill_color=red,rec,
      color=blue,explicit(a(n),n,0,11))$
```



Sequences

Sequence (real numbers) is each sequence $\{a_n\}_{n=1}^{\infty}$, whose members are real numbers $a_n \in R$ (i.e. display $N \to R$).

- **Explicit specification** (general expression) of member a_n as a function of the variable n.
- **Recursive** for the first member and a_n with the previous members.

$\{a_n\}_{n=1}^{\infty} = \{2n-1\}_{n=1}^{\infty} = \{1,3,5,\ldots\}.$

- Explicit entry $a_n = 2n-1,\ n \in N.$
- Recursive entry $a_1 = 1,\ a_{n+1} = a_n + 2,\ n \in N.$

```
(%i3) a(n):=2*n-1$ S:makelist(a(n),n,1,7);
(S)   [1,3,5,7,9,11,13]
(%i4) an:1$ (for n:1 thru 7 do (print(an),an:an+2))$
      1
      3
      5
      7
      9
      11
      13
```

The sequence $\{a_n\}_{n=1}^{\infty}$ is called:

- **Bounded from below**, if $a \in R$ exists such that $a \le a_n$ holds for all $n \in N$.

- **Bounded from above**, if $a \in R$ exists such that $a_n \leq a$ holds for all $n \in N$.
- **Bounded**, if it is bounded from below and above.

The sequence $\{a_n\}_{n=1}^{\infty}$ is called:

- **Unbounded from below**, if it is not bounded from below.
- **Unbounded from above**, if it is not bounded from above.
- **Unbounded**, if it is not bounded,
  i.e. it is not bounded from below or is not bounded from above.

α A	alfa	a	η H	éta	é	ν N	ný	n	τ T	tau	t
β B	beta	b	ϑ Θ	théta	th	ξ Ξ	ksí (xí)	x	υ Υ	ypsilon	y
γ Γ	gama	g	ι I	ióta	i	o O	omikron	o	φ Φ	fí	f
δ Δ	delta	d	κ K	kappa	k	π Π	pí	p	χ X	chí	ch
ε E	epsilon	e	λ Λ	lambda	l	ρ P	ró	r	ψ Ψ	psí	ps
ζ Z	dzéta	dz	μ M	mí	m	σ Σ	sigma	s	ω Ω	omega	ó

The sequence $\{a_n\}_{n=1}^{\infty}$ is called **monotone**:

- **Increasing**, if $a_n < a_{n+1}$ holds for all $n \in N$.
- **Decreasing**, if $a_n > a_{n+1}$ holds for all $n \in N$.

} **Strictly monotone**.

- **Nondecreasing** if $a_n \leq a_{n+1}$ holds for all $n \in N$.
- **Nonincreasing** if $a_n \geq a_{n+1}$ holds for all $n \in N$.
- **Stacionary (constant)**, if $a_n = a$ holds for all $n \in N$.

$\{k_n\}_{n=1}^{\infty}$ is an strictly increasing sequence of natural numbers.
$\Rightarrow$ $\{a_{k_n}\}_{n=1}^{\infty}$ is called **subsequence (selected sequence)** of $\{a_n\}_{n=1}^{\infty}$.

$\{a_n\}_{n=1}^{\infty} = \{2n-1\}_{n=1}^{\infty} = \{1, 3, 5, 7, 9, 11, 13, \ldots\}$.

Subsequences are e.g.

- $\{a_{k_n}\}_{n=1}^{\infty} = \{a_{2n}\}_{n=1}^{\infty} = \{a_2, a_4, a_6, \ldots\} = \{3, 7, 11, \ldots\} = \{4n-1\}_{n=1}^{\infty}$.
- $\{2n-1\}_{n=1}^{\infty}$.
- $\{2n-1\}_{n=2}^{\infty}$.
- $\{101, 109, 235, 637, \ldots\}$.

```
(%i2) a(n):=2*n-1$ makelist(a(n),n,1,7);
(%o2) [1, 3, 5, 7, 9, 11, 13]
(%i3) makelist(a(2*n),n,1,7);
(%o3) [3, 7, 11, 15, 19, 23, 27]
(%i4) makelist(a(2*n),n,2,7);
(%o4) [7, 11, 15, 19, 23, 27]
```

```
(%i5) print(a(51),a(55),a(118),a(319))$
      101 109 235 637
```

$a \in R^* = R \cup \{\pm\infty\}$ is called **accumulation value of sequence** $\{a_n\}_{n=1}^{\infty}$,
if for every neighborhood $O(a)$ there are an infinite number of members $a_n \in O(a)$.

Each sequence $\{a_n\}_{n=1}^{\infty}$ has at least one accumulation value.

Let the symbol E denote the set of all accumulation values of the sequence $\{a_n\}_{n=1}^{\infty}$.

- $\sup E = \limsup_{n\to\infty} a_n$ is called **limes superior (upper limit)** $\{a_n\}_{n=1}^{\infty}$.
- $\inf E = \liminf_{n\to\infty} a_n$ is called **limes inferior (lower limit)** $\{a_n\}_{n=1}^{\infty}$.
- $\inf E = \sup E = \lim_{n\to\infty} a_n$ (E has a single element) is called **limit** $\{a_n\}_{n=1}^{\infty}$.

- $\limsup_{n\to\infty} a_n$, $\liminf_{n\to\infty} a_n$ always exist.
- $\lim_{n\to\infty} a_n$ may not exist. If a limit exists, then it is the only one.

- $\lim_{n\to\infty} a_n = a \in R$ (finite exists).
  $\Rightarrow$ $\{a_n\}_{n=1}^{\infty}$ **converges to the number** a,
  labeling $\{a_n\}_{n=1}^{\infty} \longrightarrow a$.

  } $\{a_n\}_{n=1}^{\infty}$ **converges**, labeling $\{a_n\}_{n=1}^{\infty} \longrightarrow$.

- $\lim_{n\to\infty} a_n = \pm\infty$ (infinite exists).
  $\Rightarrow$ $\{a_n\}_{n=1}^{\infty}$ **diverges to** $\pm\infty$,
  labeling $\{a_n\}_{n=1}^{\infty} \longrightarrow \pm\infty$.
- $\lim_{n\to\infty} a_n$ does not exist. $\Rightarrow$ $\{a_n\}_{n=1}^{\infty}$ **oscillates**.

  } $\{a_n\}_{n=1}^{\infty}$ **diverges**, labeling $\{a_n\}_{n=1}^{\infty} \not\longrightarrow$.

$\{a_n\}_{n=1}^{\infty} \longrightarrow$. $\Rightarrow$ • $\{a_n\}_{n=1}^{\infty}$ is bounded.

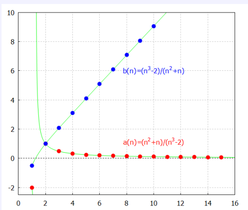
- **Finite count of Members** does not affect convergence, resp. divergence of sequence.

- $\lim_{n\to\infty} \frac{n^2+n}{n^3-2} = \lim_{n\to\infty} \frac{n^3(n^{-1}+n^{-2})}{n^3(1-2n^{-3})} = \lim_{n\to\infty} \frac{n^{-1}+n^{-2}}{1-2n^{-3}} = \frac{0+0}{1-0} = 0.$

- $\lim_{n\to\infty} \frac{n^3-2}{n^2+n} = \lim_{n\to\infty} \frac{n^2(n-2n^{-2})}{n^2(1+n^{-1})} = \lim_{n\to\infty} \frac{n-2n^{-2}}{1+n^{-1}} = \frac{\infty-0}{1+0} = \infty.$

(%i1)
```
a(n):=(n^2+n)/(n^3-2)$ Sa:makelist([n,a(n)],n,1,15)$
b(n):=(n^3-2)/(n^2+n)$ Sb:makelist([n,b(n)],n,1,15)$
print("limit  a(n)=",limit(a(n),n,inf),"  limit b(n)=",limit(b(n),n,inf))$
draw2d(grid=true,xaxis=true,yaxis=true,xrange=[0,16],yrange=[-2.5,10],
color=green,explicit(a(n),n,1,16),point_type=7,color=red,points(Sa),
  label(["a(n)=(n^2+n)/(n^3-2)",10,a(10)+1]),
color=green,explicit(b(n),n,1,16),point_type=7,color=blue,points(Sb),
  label(["b(n)=(n^3-2)/(n^2+n)",10,6]))$
```

(%o1) $limita(n) = 0 \quad limitb(n) = \infty$



- $\lim\limits_{n\to\infty} \frac{n^4-n^3}{n^2+n} = \lim\limits_{n\to\infty} \frac{n^4(1-n^{-1})}{n^2(1+n^{-1})} = \lim\limits_{n\to\infty} n^2 \cdot \lim\limits_{n\to\infty} \frac{1-n^{-1}}{1+n^{-1}} = \infty \cdot \frac{1-0}{1+0} = \infty \cdot 1 = \infty.$

- $\lim\limits_{n\to\infty} \frac{n^2+n}{n^2-2} = \lim\limits_{n\to\infty} \frac{n^2(1+n^{-1})}{n^2(1-2n^{-2})} = \lim\limits_{n\to\infty} \frac{1+n^{-1}}{1-2n^{-2}} = \frac{1+0}{1-0} = 1.$

- $$\lim_{n\to\infty} n^q = \begin{cases} \infty^q = \infty. & \Rightarrow \bullet\ \infty \text{ pre } q > 0. \\ \lim\limits_{n\to\infty} n^0 = \lim\limits_{n\to\infty} 1 = 1. & \Rightarrow \bullet\ 1 \text{ pre } q = 0. \\ \lim\limits_{n\to\infty} \frac{1}{n^{-q}} = \frac{1}{\lim\limits_{n\to\infty} n^{-q}} = \frac{1}{\infty} = 0. & \Rightarrow \bullet\ 0 \text{ pre } q < 0\ (-q > 0). \end{cases}$$

Geometric Sequence

- $$\lim_{n\to\infty} q^n = \begin{cases} q^n \to \infty. & \Rightarrow \bullet\ \infty \text{ for } q > 1. \\ q^n = 1^n = 1. & \Rightarrow \bullet\ 1 \text{ for } q = 1. \\ q^n \to 0. & \Rightarrow \bullet\ 0 \text{ for } q \in (0;1). \\ q^n = q^{2k} = 1,\ q^n = q^{2k+1} = -1. & \Rightarrow \bullet\ \nexists \text{ for } q = -1. \\ q^n = q^{2k} \to \infty,\ q^n = q^{2k+1} \to -\infty. & \Rightarrow \bullet\ \nexists \text{ for } q < -1. \end{cases}$$

- The number e is called **Euler's number**. Its value is approximately 2.718 281 827.

$a_n > 0,\ n \in N. \ \Rightarrow$ • $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = \lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n}$ (if limits exist).

$a_n \geq 0,\ n \in N,\ \lim\limits_{n\to\infty} \sqrt[n]{a_n} = a \in R^*. \ \Rightarrow$ • $\lim\limits_{n\to\infty} a_n = \begin{cases} 0 & \text{for } a < 1, \\ \infty & \text{for } a > 1. \end{cases}$

$a_n > 0,\ n \in N,\ \lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = a \in R^*. \ \Rightarrow$ • $\lim\limits_{n\to\infty} a_n = \begin{cases} 0 & \text{for } a < 1, \\ \infty & \text{for } a > 1. \end{cases}$

Dôležité limity. Important limits.

- $\lim\limits_{n\to\infty} \sqrt[n]{a} = 1$ for $a > 0$.
- $\lim\limits_{n\to\infty} \sqrt[n]{n!} = \infty$.
- $\lim\limits_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \mathrm{e}$.
- $\lim\limits_{n\to\infty} n\left(\sqrt[n]{\mathrm{e}} - 1\right) = 1$.
- $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$.
- $\lim\limits_{n\to\infty} \frac{n!}{n^n} = 0$.
- $\lim\limits_{n\to\infty} \left(1 + \frac{b}{n}\right)^n = \mathrm{e}^b$ for $b \in R$.
- $\lim\limits_{n\to\infty} n\left(\sqrt[n]{a} - 1\right) = \ln a$ for $a > 0$.

Infinite Series

Infinite series are closely related to sequences and generalize the concept additions to an infinite number of additions. Simple examples are fractions and periodic numbers.

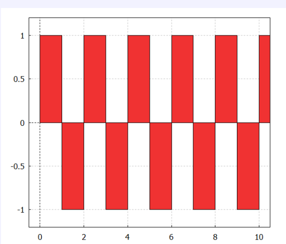
$\{a_n\}_{n=1}^{\infty}$ is a sequence.

$$\Rightarrow \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$ is called **(infinite numeric) series**.

Some rules applicable to finite counts do not apply to infinite series.
Does not apply e.g. associative law:

$$\sum_{n=1}^{\infty} (-1)^{n+1} = \begin{cases} (1-1)+(1-1)+(1-1)+\cdots = 0+0+0+\cdots = 0, \\ 1+(-1+1)+(-1+1)+\cdots \quad = 1+0+0+\cdots = 1. \end{cases}$$

(%i1)
```
a(n):=(-1)^(n+1)$ rec:makelist(rectangle([i-1,0],[i,a(i)]),i,1,11)$
draw2d(grid=true,xaxis=true,yaxis=true,xrange=[-.5,10.5],yrange=[-1.2,1.2],
border=true,color=black,fill_color=red,rec)$
```



$\sum\limits_{n=1}^{\infty} a_n$ is a series.

- $s_k = \sum\limits_{i=1}^{k} a_i = a_1 + a_2 + \cdots + a_k$, $k \in N$ is called **kth partial sum of series** $\sum\limits_{n=1}^{\infty} a_n$.
- $r_k = \sum\limits_{i=k+1}^{\infty} a_k = a_{k+1} + a_{k+2} + a_{k+3} + \cdots$ is called **kth rest of the series** $\sum\limits_{n=1}^{\infty} a_n$.
- $\{s_k\}_{n=1}^{\infty} = \{s_n\}_{n=1}^{\infty}$ is called **sequence of partial sums of series** $\sum\limits_{n=1}^{\infty} a_n$.

> The relationship between $\sum\limits_{n=1}^{\infty} a_n$ and the sequence $\{s_n\}_{n=1}^{\infty}$ is mutually unique.

For $\{s_n\}_{n=1}^{\infty}$ and $\sum\limits_{n=1}^{\infty} a_n$:

- $s_1 = a_1$.
- $s_2 = a_1 + a_2 = s_1 + a_2$.
- $s_3 = a_1 + a_2 + a_3 = s_2 + a_3$.

  $\ldots$
- $s_n = a_1 + a_2 + \cdots + a_{n-1} + a_n = s_{n-1} + a_n$.

- $a_1 = s_1 = s_1 - s_0$, where $s_0 = 0$.
- $a_2 = s_2 - s_1$.
- $a_3 = s_3 - s_2$.
- $a_n = s_n - s_{n-1}$, $n \in N$.

The sum of the series $\sum\limits_{n=1}^{\infty} a_n$ is called $\lim\limits_{n\to\infty} s_n = s \in R^*$ (if any), labeling $\sum\limits_{n=1}^{\infty} a_n = s$.

- $\lim\limits_{n\to\infty} s_n = s \in R$ (finite exists).

  $\Rightarrow$ $\sum\limits_{n=1}^{\infty} a_n$ **converges to the sum** s,

  labeling $\sum\limits_{n=1}^{\infty} a_n \longrightarrow s$, resp. $\sum\limits_{n=1}^{\infty} a_n = s$.

  } $\sum\limits_{n=1}^{\infty} a_n$ **converges**, labeling $\sum\limits_{n=1}^{\infty} a_n \longrightarrow$.

- $\lim\limits_{n\to\infty} s_n = \pm\infty$ (infinite exists).

  $\Rightarrow$ $\sum\limits_{n=1}^{\infty} a_n$ **diverges to** $\pm\infty$,

  labeling $\sum\limits_{n=1}^{\infty} a_n \longrightarrow \pm\infty$, resp. $\sum\limits_{n=1}^{\infty} a_n = \pm\infty$.

- $\lim\limits_{n\to\infty} s_n$ does not exist.

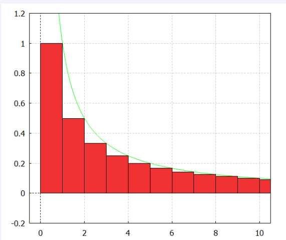
  $\Rightarrow$ $\sum\limits_{n=1}^{\infty} a_n$ **oscillates (has no sum)**.

  } $\sum\limits_{n=1}^{\infty} a_n$ **diverges**, labeling $\sum\limits_{n=1}^{\infty} a_n \not\longrightarrow$.

Harmonic Series

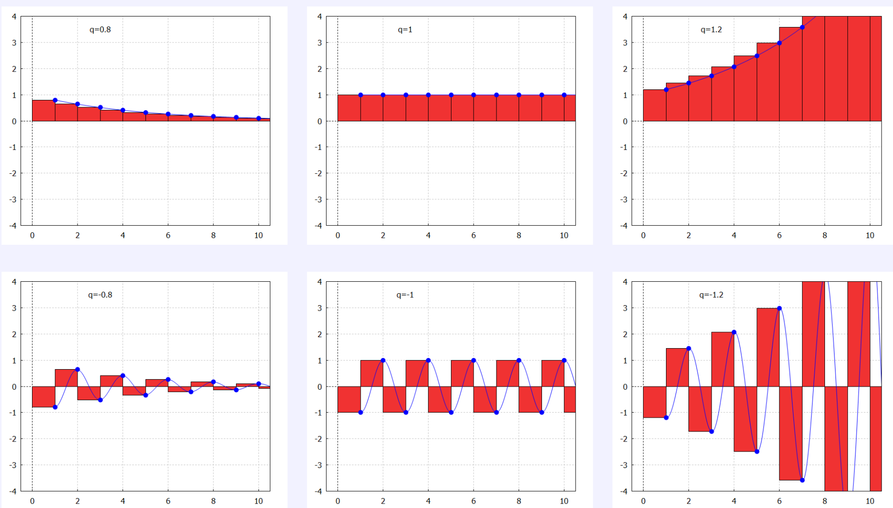
$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots = \infty.$$

(%i1)
```
a(n):=1/n$ rec:makelist(rectangle([i-1,0],[i,a(i)]),i,1,11)$
draw2d(grid=true,xaxis=true,yaxis=true,xrange=[-.5,10.5],yrange=[-.2,1.2],
color=green,explicit(a(n),n,.5,11),
border=true,color=black,fill_color=light_red,rec)$
```



In the following example, just change the value of q at the beginning.

(%i1)
```
q:0.8$ a(n,q):=q^n$ peca:makelist([i,a(i,q)],i,1,11)$
reca:makelist(rectangle([i-1,0],[i,a(i,q)]),i,1,11)$
draw2d(grid=true,xaxis=true,yaxis=true,xrange=[-.5,10.5],yrange=[-4,4],
border=true,color=black,fill_color=light_red,reca,
label([concat("q=",string(q)),3,3.5]), color=blue,explicit(a(n,q),n,1,11),
point_type=7,color=blue,points(peca))$
```



Geometric series

$\sum\limits_{n=1}^{\infty} q^{n-1} = 1 + q + q^2 + \cdots = \frac{1}{1-q}$ for all $q \in (-1;1)$.

$$\sum_{n=1}^{\infty} q^{n-1},\ s_n = 1 + q + \cdots + q^{n-1} = (q^{n-1} + \cdots + q + 1)\frac{q-1}{q-1} = \frac{q^n-1}{q-1} = \frac{q^{n-1}-\frac{1}{q}}{1-\frac{1}{q}}.$$

- $\sum\limits_{n=1}^{\infty} q^{n-1} = \lim\limits_{n\to\infty} s_n = \begin{cases} \lim\limits_{n\to\infty} \frac{q^n-1}{q-1} = \frac{\infty-1}{q-1} = \infty. & \Rightarrow \bullet\ \infty \quad \text{for } q > 1. \\ \lim\limits_{n\to\infty} n = \infty. & \Rightarrow \bullet\ \infty \quad \text{for } q = 1. \\ \lim\limits_{n\to\infty} \frac{q^n-1}{q-1} = \frac{0-1}{q-1} = \frac{1}{1-q}. & \Rightarrow \bullet\ \frac{1}{1-q} \quad \text{for } q \in (-1;1). \\ -1+1-1+1-1+\cdots. & \Rightarrow \bullet\ \nexists \quad \text{for } q = -1. \\ \left.\begin{array}{l} \frac{q^{2k-1}-\frac{1}{q}}{1-\frac{1}{q}} = \frac{-\infty-\frac{1}{q}}{1-\frac{1}{q}} = -\infty \text{ for } n = 2k. \\ \frac{q^{2k+1-1}-\frac{1}{q}}{1-\frac{1}{q}} = \frac{\infty-\frac{1}{q}}{1-\frac{1}{q}} = \infty \text{ for } n = 2k+1. \end{array}\right\} & \Rightarrow \bullet\ \nexists \quad \text{for } q < -1. \end{cases}$

```
(%i4) sq(q):=sum(q^n,n,1,inf)$
      sq(1/2),simpsum; sq(1/3),simpsum; sq(-1/2),simpsum; sq(2),simpsum;
```

(%i1) 1
(%i2) $\frac{1}{2}$
(%i3) $-\frac{1}{3}$
(%i4) *sum: sum is divergent.*

Necessary condition of series convergence

$\sum\limits_{n=1}^{\infty} a_n$ converges. $\Rightarrow$ • $\lim\limits_{n\to\infty} a_n = 0$.

- $\sum\limits_{n=1}^{\infty} a_n$ converges. $\Rightarrow \sum\limits_{n=1}^{\infty} a_n = \lim\limits_{n\to\infty} s_n = s \in R$.

$$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} (s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0.$$

Not valid $\lim\limits_{n\to\infty} a_n = 0. \Rightarrow$ • $\sum\limits_{n=1}^{\infty} a_n \not\longrightarrow$ (oscillates or diverges to $\pm\infty$).

- **Finite count of Members** does not affect convergence, resp. divergence work.
- **Finite count of members** affects the sum of the series.

Series with Nonnegative Members

- $\sum\limits_{n=1}^{\infty} a_n$ with nonnegative members ($a_n \geq 0,\ n \in N$) always has a sum.

$0 \leq a_n \leq b_n,\ n \in N. \ \Rightarrow$ • $0 \leq \sum\limits_{n=1}^{\infty} a_n \leq \sum\limits_{n=1}^{\infty} b_n \leq \infty.$

Comparison Criterion

$0 \leq a_n \leq b_n,\ n \in N. \ \Rightarrow$ • $\sum\limits_{n=1}^{\infty} b_n \longrightarrow. \quad \Rightarrow$ • $\sum\limits_{n=1}^{\infty} a_n \longrightarrow.$

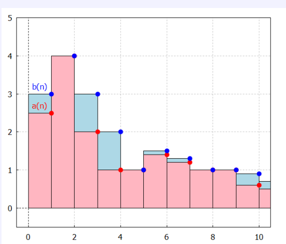
• $\sum\limits_{n=1}^{\infty} a_n \longrightarrow \infty. \ \Rightarrow$ • $\sum\limits_{n=1}^{\infty} b_n \longrightarrow \infty.$

Limit form

$0 < a_n \leq b_n,\ n \in N,\ \lim\limits_{n\to\infty} \frac{a_n}{b_n} \in (0;\infty). \ \Rightarrow$ • $\sum\limits_{n=1}^{\infty} b_n \longrightarrow. \quad \Leftrightarrow$ • $\sum\limits_{n=1}^{\infty} a_n \longrightarrow.$

• $\sum\limits_{n=1}^{\infty} a_n \longrightarrow \infty. \ \Leftrightarrow$ • $\sum\limits_{n=1}^{\infty} b_n \longrightarrow \infty.$

(%i1)
```
a:[2.5,4,2,1,1,1.4,1.2,1,1,0.6,0.5]$ pa:makelist([i,a[i]],i,1,11)$
ra:makelist(rectangle([i-1,0],[i,a[i]]),i,1,11)$
b:[3.0,4,3,2,1,1.5,1.3,1,1,0.9,0.7]$ pb:makelist([i,b[i]],i,1,11)$
rb:makelist(rectangle([i-1,0],[i,b[i]]),i,1,11)$
draw2d(grid=true,xaxis=true,yaxis=true,xrange=[-.5,10.5],yrange=[-.5,5],
border=true,color=black,fill_color=light_blue,
rb,color=black,fill_color=light_pink,ra,
point_type=7,color=red,points(pa),point_type=7,color=blue,points(pb),
color=red,label(["a(n)",.5,2.7]),color=blue,label(["b(n)",.5,3.2]))$
```



Ratio d'Alembert criterion

$a_n > 0,\ n \in N. \quad \Rightarrow \quad \bullet\ \frac{a_{n+1}}{a_n} \leq q < 1,\ q \in (0;1),\ n \in N. \ \Rightarrow \ \bullet \sum_{n=1}^{\infty} a_n \longrightarrow.$

$\bullet\ 1 \leq \frac{a_{n+1}}{a_n},\ n \in N. \quad \Rightarrow \ \bullet \sum_{n=1}^{\infty} a_n \longrightarrow \infty.$

Limit form

$a_n > 0,\ n \in N,\ \lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = p. \quad \Rightarrow \quad \bullet\ p < 1. \ \Rightarrow \ \bullet \sum_{n=1}^{\infty} a_n \longrightarrow.$

$\bullet\ p > 1. \ \Rightarrow \ \bullet \sum_{n=1}^{\infty} a_n \longrightarrow \infty.$

For $p = 1$ we cannot decide on the convergence or divergence of the series $\sum\limits_{n=1}^{\infty} a_n$.

Root Cauchy Criterion

$a_n \geq 0,\ n \in N. \quad \Rightarrow \quad \bullet\ \sqrt[n]{a_n} \leq q < 1,\ q \in (0;1),\ n \in N. \ \Rightarrow \ \bullet \sum_{n=1}^{\infty} a_n \longrightarrow.$

$\bullet\ 1 \leq \sqrt[n]{a_n},\ n \in N. \quad \Rightarrow \ \bullet \sum_{n=1}^{\infty} a_n \longrightarrow \infty.$

Limit form

$a_n \geq 0,\ n \in N,\ \lim\limits_{n\to\infty} \sqrt[n]{a_n} = p. \quad \Rightarrow \quad \bullet\ p < 1. \ \Rightarrow \ \bullet \sum_{n=1}^{\infty} a_n \longrightarrow.$

$\bullet\ p > 1. \ \Rightarrow \ \bullet \sum_{n=1}^{\infty} a_n \longrightarrow \infty.$

For $p = 1$ we cannot decide on the convergence or divergence of the series $\sum\limits_{n=1}^{\infty} a_n$.

$$\left.\begin{array}{l} \bullet\ \lim\limits_{n\to\infty} \frac{a^{n+1}}{(n+1)!}\,\frac{n!}{a^n} = \lim\limits_{n\to\infty} \frac{a}{n+1} = \frac{a}{\infty} = 0 < 1. \\ \bullet\ \lim\limits_{n\to\infty} \sqrt[n]{\frac{a^n}{n!}} = \lim\limits_{n\to\infty} \frac{a}{\sqrt[n]{n!}} = \frac{a}{\infty} = 0 < 1. \end{array}\right\} \Rightarrow \ \bullet \sum_{n=1}^{\infty} \frac{a^n}{n!} \longmapsto \text{for } a > 0.$$

- $\lim\limits_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!}\,\frac{n!}{n^n} = \lim\limits_{n\to\infty} \frac{(n+1)^n}{n^n} = \lim\limits_{n\to\infty} \left(1+\frac{1}{n}\right)^n = \mathrm{e} > 1.$ $\Rightarrow$ • $\sum\limits_{n=1}^{\infty} \frac{n^n}{n!} \longmapsto \infty.$

```
(%i4) an(n):=n^n/n!$ limit(an(n),n,inf,plus);
      limit(an(n+1)/an(n),n,inf,plus);
      limit((an(n))^(1/n),n,inf,plus);
(%o2) ∞
(%o3) e
(%o8) e
(%i9) an(n,a):=a^n/n!$ a:2$ limit(an(n,a),n,inf,plus);
      limit(an(n+1,a)/an(n,a),n,inf,plus);
      limit((an(n,a))^(1/n),n,inf,plus);
(%i7) 0
(%o8) 0
(%o9) 0
```

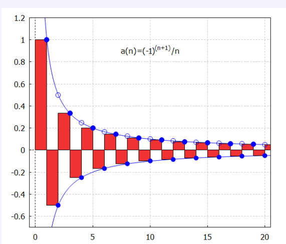
Absolute, relative convergence and alternating series

$\sum\limits_{n=1}^{\infty} a_n$ is a series.

- $\sum\limits_{n=1}^{\infty} |a_n| \longrightarrow.$ $\Rightarrow$ $\sum\limits_{n=1}^{\infty} a_n$ **converges absolutely**, labeling $\sum\limits_{n=1}^{\infty} a_n \xrightarrow{\mathrm{A}}.$
- $\sum\limits_{n=1}^{\infty} a_n \longrightarrow,$ $\sum\limits_{n=1}^{\infty} |a_n| \longrightarrow \infty.$

$\Rightarrow$ $\sum\limits_{n=1}^{\infty} a_n$ **converges relative (non-absolutely)**, labeling $\sum\limits_{n=1}^{\infty} a_n \xrightarrow{\mathrm{R}}.$

```
(%i1) a(n):=(-1)^(n+1)/n$ pa:makelist([i,a(i)],i,1,21)$
      ra:makelist(rectangle([i-1,0],[i,a(i)]),i,1,21)$
      draw2d(grid=true,xaxis=true,yaxis=true,xrange=[-.5,20.5],yrange=[-.7,1.2],
      color=blue,explicit(abs(a(n)),n,.5,21),explicit(-abs(a(n)),n,.5,21),
      border=true,color=black,fill_color=light_red,ra,
      label(["a(n)=(-1)^{(n+1)}/n",10,.9]),
      point_type=6,color=blue,points(abs(pa)),point_type=7,color=blue,points(pa))$
```



$\sum_{n=1}^{\infty} a_n \xrightarrow{\text{A}}$ (converges absolutely). $\Rightarrow$ • $\sum_{n=1}^{\infty} a_n \longrightarrow$ (converges).

Leibniz's criterion

$$\left.\begin{array}{l} a_n \geq 0,\ n \in N,\ \{a_n\}_{n=1}^{\infty} \text{ is nonincreasing.} \\ \lim_{n\to\infty} a_n = 0. \end{array}\right\} \Rightarrow \bullet \sum_{n=1}^{\infty} (-1)^{n+1} a_n \longrightarrow.$$

Anharmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n+1}}{n} + \cdots \xrightarrow{\text{R}} \ln 2.$

Functions

Funkcia $y = f(x)$, $x \in D(f)$, i.e. $f\colon D(f) \to H(f)$.

- Set $\{[x; y] \in R^2;\ x \in D(f),\ y = f(x)\}$ is called **graph of function f**.
- **Function of a real variable**, if the domain $D(f) \subset R$.
- **Real Function**, if the range of values $H(f) \subset R$.

$y = f(x)$, $x \in A$ is called:

- **Injective**, if for all $x_1, x_2 \in A$, $x_1 \neq x_2$ holds $f(x_1) \neq f(x_2)$,
  i.e. the equality $f(x_1) = f(x_2)$ implies the equality $x_1 = x_2$.
- **Surjective**, if $f(A) = B$,
  i.e. for every $y \in B$ there exists $x \in A$ also that $y = f(x)$.
- **Bijective**, if it is injective and surjective.

$y = f(x)$, $x \in D(f)$ sa vyjadruje:

- **Explici**, i.e. analytically by the formula $y = f(x)$, $x \in D(f)$.
- **Parametric** equations $x = \varphi(t)$, $y = \psi(t)$, $t \in J$, $J \subset R$, where $\varphi, \psi\colon J \to R$.
  The t parameter has an auxiliary meaning.
- **Implicit** equation $F(x, y) = 0$, where $F\colon R^2 \to R$ and conditions for $[x; y]$.

If we want to display the function specified by default in Maxima, we have to load library `implicit_plot`.

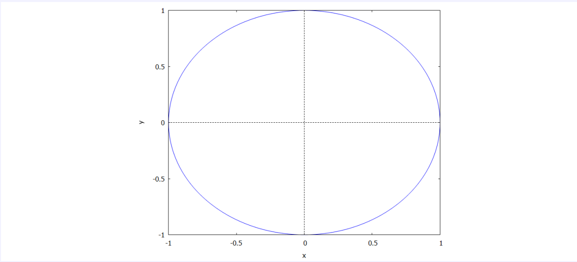
```
(%i1) load(implicit_plot);
```

(%o1) *../share/contrib/implicit_plot.lisp*

```
(%i2) implicit_plot(x^2+y^2-1, [x,-1,1], [y,-1,1])$
```

implicit_plot is now obsolete. Using plot2d instead:

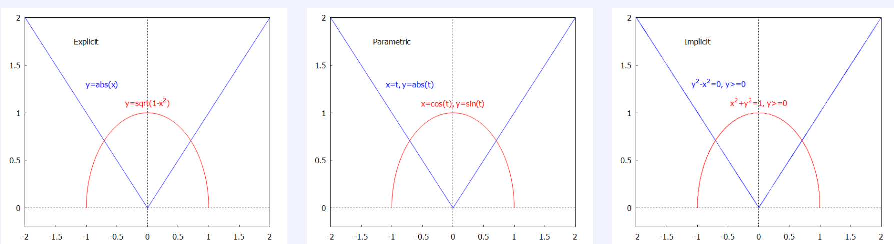
```
    plot2d (y^2+x^2-1 = 0, [x,-1,1], [y,-1,1])
(%i2) plot2d(x^2+y^2-1=0, [x,-1,1], [y,-1,1])$  /* is correct */
```



Function f: $y = |x|$, $x \in R$ can be entered eg:

- Explicit: $y = \sqrt{x^2}$, resp. $y = \max\{-x, x\}$.
- Parametric: $x = t$, $y = |t|$ $t \in R$, resp. $x = t$, $y = \sqrt{t^2}$, $t \in R$.
- Implicit: $y^2 - x^2 = 0$, $y \geq 0$, resp. $y - |x| = 0$.

```
(%i1) load(implicit_plot)$
(%i2) draw2d(xaxis=true,yaxis=true,xrange=[-2,2],yrange=[-.2,2],
      color=blue,explicit(abs(x),x,-2,2),label(["y=abs(x)",-.75,1.3]),
      color=red,explicit(sqrt(1-x^2),x,-1,1),label(["y=sqrt(1-x^2)",0,1.1]),
      color=black,label(["Explicit",-1,1.75]))$
(%i3) draw2d(xaxis=true,yaxis=true,xrange=[-2,2],yrange=[-.2,2],
      color=blue,parametric(t,abs(t),t,-2,2),label(["x=t, y=abs(t)",-.7,1.3]),
      color=red,nticks=100,parametric(cos(t),sin(t),t,0,%pi),
      label(["x=cos(t), y=sin(t)",0,1.1]),
      color=black,label(["Parametric",-1,1.75]))$
(%i4) draw2d(xaxis=true,yaxis=true,xrange=[-2,2],yrange=[-.2,2],
      color=blue,implicit(y^2-x^2,x,-2,2,y,0,2),label(["y^2-x^2=0, y>=0",-.65,1.3]),
      color=red,implicit(x^2+y^2-1,x,-1,1,y,0,1),label(["x^2+y^2=1, y>=0",0,1.1]),
      color=black,label(["Implicit",-1,1.75]))$
```



$y = f(x)$, $x \in D(f)$ is called **on the set** $A \subset D(f)$:

- **Bounded from below**, if $a \in R$ also exists, that $a \leq f(x)$ holds for all $x \in A$.
- **Bounded from above**, if $a \in R$ also exists, that $f(x) \leq a$ holds for all $x \in A$.
- **Bounded**, if bounded at the from below and from above of the set A,
  i.e. if there are $a_1, a_2 \in R$ such that for all $x \in A$ $a_1 \leq f(x) \leq a_2$.

$y = f(x)$, $x \in D(f)$ is called **on the set** $A \subset D(f)$:

- **Unbounded from below**, if it is not bounded from below on the set A,
- **Unbounded from above**, if it is not bounded from above on the set A,
- **Unbounded**, if it is not bounded on the set A,
  i.e. is unbounded from below or is unbounded from above.

- $A \subset D(f)$, $A \neq D(f)$. $\Rightarrow$ **Local property.**
- $A = D(f)$. $\Rightarrow$ **Global Property** on the whole $D(f)$.
  Words on the set $D(f)$ are usually omitted.

$y = f(x)$, $x \in D(f)$, $A \subset D(f)$:

- $\inf\limits_{x \in A} f(x) = \inf \{f(x); x \in A\} = \inf f(A)$ is called **infimum f on set A**.
- $\sup\limits_{x \in A} f(x) = \sup \{f(x); x \in A\} = \sup f(A)$ is called **supremum f to A**.
- $\inf f(x) = \inf \{f(x); x \in D(f)\}$ is called **infimum f**.
- $\sup f(x) = \sup \{f(x); x \in D(f)\}$ is called **supremum f**.

$f\colon\ y = x^2 + 1$, $x \in R$.

- f is bounded from below, not bounded from above, not bounded.
  The minimum (global) function f is 1, the function f acquires it at the point $x = 0$, the maximum does not exist.
- f is bounded on the interval $\langle -1; 2)$.
  The local minimum of the f function on the interval $\langle -1; 2)$ is 1 and acquires it at the point $x = 0$. The local maximum does not exist, the local supreme is 5.

$y = f(x)$, $x \in D(f)$, $A \subset D(f)$, $x_0 \in A$:

- $f(x_0) = \min f(A) = \min \{f(x); x \in A\}$ is called **minimum f on the set A**.

  $$f(x_0) \begin{cases} \textbf{minimum}, & \text{if for all } x \in A \text{ holds } f(x_0) \leq f(x). \\ \textbf{strict minimum}, & \text{if for all } x \in A,\ x \neq x_0 \text{ holds } f(x_0) < f(x). \end{cases}$$

- $f(x_0) = \max f(A) = \max \{f(x); x \in A\}$ is called **maximum f on the set A**.

$$f(x_0)\begin{cases}\textbf{maximum}, & \text{if for all } x\in A \text{ holds } f(x_0)\geq f(x).\\ \textbf{strict maximum}, & \text{if for all } x\in A,\ x\neq x_0 \text{ holds } f(x_0)>f(x).\end{cases}$$

- The minimum and maximum are called **extremes**.
- The strict minimum and strict maximum are called **strict extremes**.

- $A=D(f)$. $\Rightarrow$ The extremes are called **global (absolute)**.
- $A\subset D(f)$, $A\neq D(f)$. $\Rightarrow$ The extremes are called **local (on the set A)**.

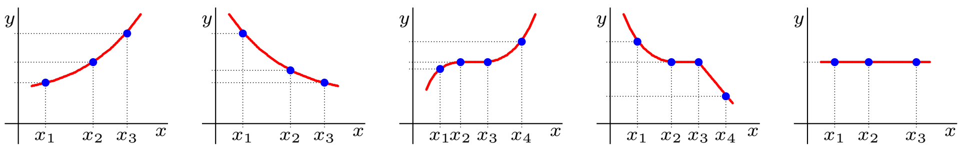
It is enough to investigate local extremes in some neighborhood $O(x_0)\subset D(f)$.

$y=f(x)$, $x\in D(f)$ is called **on the set $A\subset D(f)$ monotone**:

- **Increasing**, if for all $x_1,x_2\in A$, $x_1<x_2$ holds $f(x_1)<f(x_2)$.
- **Decreasing**, if for all $x_1,x_2\in A$, $x_1<x_2$ holds $f(x_1)>f(x_2)$.

(Increasing and decreasing functions are called **strictly monotone**.)

- **Nondecreasing**, if for all $x_1,x_2\in A$, $x_1<x_2$ holds $f(x_1)\leq f(x_2)$.
- **Nonincreasing**, if for all $x_1,x_2\in A$, $x_1<x_2$ holds $f(x_1)\geq f(x_2)$.
- **Constant**, if for all $x_1,x_2\in A$ holds $f(x_1)=f(x_2)$, i.e. $f(x_1)=c$, where $c\in R$.



Graphs of increasing, decreasing, nondecreasing, nonincreasing and constant function

$y=f(x)$, $x\in D(f)$ is called:

- **Even**, if for all $x\in D(f)$ holds $-x\in D(f)$, $f(x)=f(-x)$.
- **Odd**, if for all $x\in D(f)$ holds $-x\in D(f)$, $f(x)=-f(-x)$.

$y=f(x)$, $x\in D(f)$ is called:

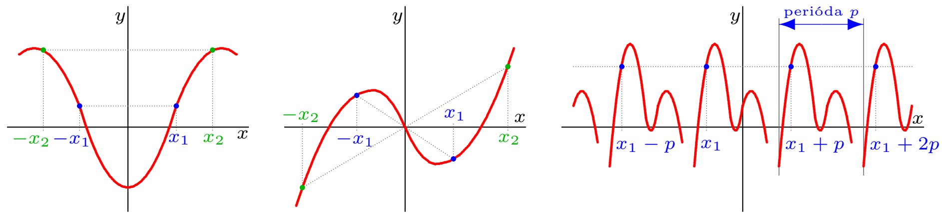
- **Periodic**, if $p\in R$, $p\neq 0$ exists such that for all $x\in D(f)$ hold $x+p\in D(f)$, $x-p\in D(f)$, $f(x)=f(x+p)=f(x-p)$.

  The number p is called **period**.

  The smallest $p>0$ (if exists) is called **primitive (base) period**.

  Each integer multiple of a period is also a period.

You only need to investigate the function on an interval of length p (**periodicity interval**).



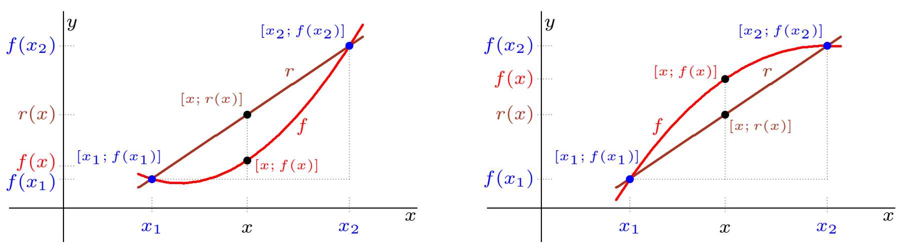
Graph of even, odd and periodic function

$y = f(x)$, $x \in D(f)$, $I \subset D(f)$ is the interval f is called **on the interval** I:

- **Convex**, if for all $x, x_1, x_2 \in I$, $x_1 < x < x_2$ holds $f(x) \leq r(x)$.
- **Strictly convex**, if for all $x, x_1, x_2 \in I$, $x_1 < x < x_2$ holds $f(x) < r(x)$.
- **Concave**, if for all $x, x_1, x_2 \in I$, $x_1 < x < x_2$ holds $f(x) \geq r(x)$.
- **Strictly concave**, if for all $x, x_1, x_2 \in I$, $x_1 < x < x_2$ holds $f(x) > r(x)$.

The line $y = r(x)$ connects the points $[x_1; f(x_1)]$ and $[x_2; f(x_2)]$.

- f is convex on the interval $I \subset D(f)$.
  $\Leftrightarrow$ • $f(px_1 + qx_2) \leq pf(x_1) + qf(x_2)$ for all $x_1, x_2 \in I$, $p \in (0; 1)$, $q = 1 - p$.
- f is concave on the interval $I \subset D(f)$.
  $\Leftrightarrow$ • $f(px_1 + qx_2) \geq pf(x_1) + qf(x_2)$ for all $x_1, x_2 \in I$, $p \in (0; 1)$, $q = 1 - p$.



Convex and concave function

$y = f(x)$, $x \in D(f)$, poind $x_0 \in D(f)$ is called:

- **Inflection point** f, if there exists a neighborhood $O(x_0)$ such that

  in $O^-(x_0)$ is f strictly convex, in $O^+(x_0)$ is f strictly concave, resp. in $O^-(x_0)$ is f strictly concave, in $O^+(x_0)$ is f strictly convex.

- **Zero point (root) f** if $f(x_0) = 0$.

$y = f(x),\ x \in D(f),\ A \subset D(f)$.

- $y = h(x),\ x \in A$ is called **restriction) f to set A**, labeling $h = f|_A$.

$y = f(x),\ x \in D(f),\ y = g(x),\ x \in D(g),\ H(f) \subset D(g)$.

- $y = F(x) = g[f(x)],\ x \in D(f)$ is called **function composition f a g**.
  f is called **inner function**, g is called **outer function**.

$y = f(x),\ x \in D(f) \to H(f)$, i.e. $y = f(x)\colon D(f) \to H(f)$.

- $x = g(y)\colon H(f) \to D(f)$ such that $[y;x] \in g \ \Leftrightarrow\ [x;y]\,\inf$, i.e. $x = g(y) \ \Leftrightarrow\ y = f(x)$, is called **inverse function to f**, labeling $g = f^{-1}$.

$f\colon D(f) \to H(f)$ is a bijection. $\Rightarrow$ • $f^{-1}\colon H(f) \to D(f)$ exists and holds:

- f^{-1} is a bijection.
- $f[f^{-1}(y)] = y$ for all $y \in H(f) = D(f^{-1})$.
- $(f^{-1})^{-1} = f$.
- $f^{-1}[f(x)] = x$ for all $x \in D(f) = H(f^{-1})$.

Elementary Functions

Elementary functions are of great practical importance. They can be used to describe (at least approximately) many natural and social laws and phenomena.

Elementary Function is called every function created using the operations of addition, subtraction, multiplication, division and composition of the functions:

- $y=\text{const.}$, • $y=x$, • $y=\mathrm{e}^x$, • $y=\ln x$, • $y=\sin x$, • $y=\arcsin x$, • $y=\operatorname{arctg} x$.

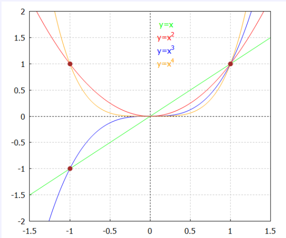
Polynomal (rational integral function) of degree n is called

$$f_n\colon\ y = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n,\ a_0, a_1, \ldots, a_n \in R,\ n \in N \cup \{0\},\ a_n \neq 0.$$

- The numbers $a_0, a_1, \ldots, a_n$ are called **coefficients**. Natural $D(f_n) = R$.
- $f_0\colon\ y = a_0,\ a_0 \neq 0$ is called **constant function**.
- $f_1\colon\ y = a_0 + a_1x,\ a_1 \neq 0$ is called **linear function**.
- $f_2\colon\ y = a_0 + a_1x + a_2x^2,\ a_2 \neq 0$ is called **quadratic function**.

```
(%i1)  draw2d(grid=true,xaxis=true,yaxis=true,xrange=[-1.5,1.5],yrange=[-2,2],
       color=green,explicit(x,x,-1.5,1.5),label(["y=x",.2,1.75]),
       color=red,explicit(x^2,x,-1.5,1.5),label(["y=x^2",.2,1.5]),
       color=blue,explicit(x^3,x,-1.5,1.5),label(["y=x^3",.2,1.25]),
       color=orange,explicit(x^4,x,-1.5,1.5),label(["y=x^4",.2,1]),
```

```
color=brown,point_type=7,points([[-1,-1],[-1,1],[1,1]])$
```

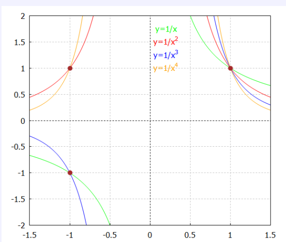


Rational polynomial function is called

$$f\colon\ y=\frac{f_n(x)}{f_m(x)}=\frac{a_0+a_1x+a_2x^2+\cdots+a_nx^n}{b_0+a_1x+a_2x^2+\cdots+b_mx^m},\ n,m\in N-\{0\}.$$

- f_n, f_m are polynomials of degrees n and m, $a_0,a_1,\ldots,a_n\in R$, $b_0,b_1,\ldots,b_m\in R$.

(%i1)
```
draw2d(grid=true,xaxis=true,yaxis=true,xrange=[-1.5,1.5],yrange=[-2,2],
color=green,explicit(1/x,x,-1.5,1.5),label(["y=1/x",.2,1.75]),
color=red,explicit(1/x^2,x,-1.5,1.5),label(["y=1/x^2",.2,1.5]),
color=blue,explicit(1/x^3,x,-1.5,1.5),label(["y=1/x^3",.2,1.25]),
color=orange,explicit(1/x^4,x,-1.5,1.5),label(["y=1/x^4",.2,1]),
color=brown,point_type=7,points([[-1,-1],[-1,1],[1,1]])$
```

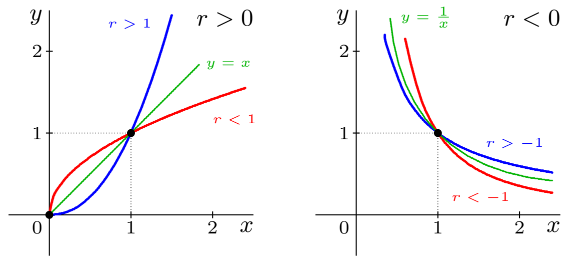


Power function is called

$$f\colon\ y=x^r,\ r\in R,\ r\neq 0.$$

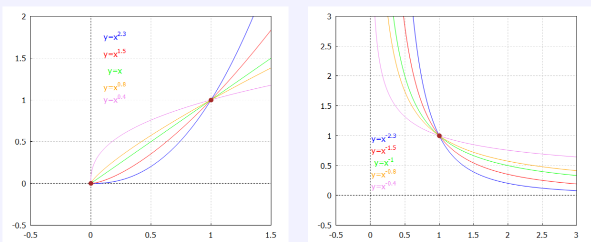
- For $r=n\in N$ is $f\colon\ y=x^n$ polynomial.
- For $r=-n\in Z^-$, $f\colon\ y=x^{-n}=\frac{1}{x^n}$ is a rational polynomial.
- For $r\neq 0$, $f^{-1}\colon\ y=x^{1/r}$.
- For $r>0$, f is increasing, natural $D(f)=\langle 0;\infty)$.

- For $r < 0$, f is decreasing, natural $D(f) = (0; \infty)$.



Function $f\colon\ y = x^r$ for $r > 0$ a $r < 0$

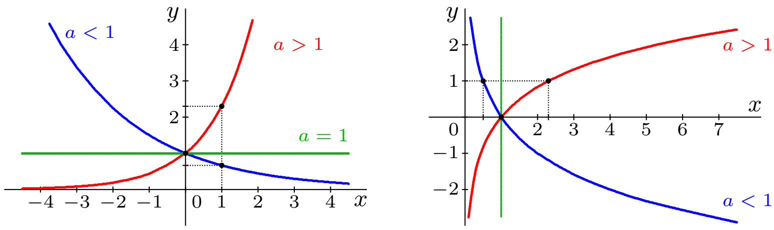
```
(%i1) draw2d(grid=true,xaxis=true,yaxis=true,xrange=[-.5,1.5],yrange=[-.5,2],
      color=blue,explicit(x^2.3,x,0,1.5),label(["y=x^{2.3}",.2,1.75]),
      color=red,explicit(x^1.5,x,0,1.5),label(["y=x^{1.5}",.2,1.55]),
      color=green,explicit(x,x,-0,1.5),label(["y=x",.2,1.35]),
      color=orange,explicit(x^.8,x,0,1.5),label(["y=x^{0.8}",.2,1.15]),
      color=violet,explicit(x^.4,x,0,1.5),label(["y=x^{0.4}",.2,1]),
      color=brown,point_type=7,points([[0,0],[1,1]]))$
(%i2) draw2d(grid=true,xaxis=true,yaxis=true,xrange=[-.5,3],yrange=[-.5,3],
      color=blue,explicit(x^-2.3,x,0,3),label(["y=x^{-2.3}",.2,.95]),
      color=red,explicit(x^-1.5,x,0,3),label(["y=x^{-1.5}",.2,.75]),
      color=green,explicit(x^-1,x,-0,3),label(["y=x^{-1}",.2,.55]),
      color=orange,explicit(x^-.8,x,0,3),label(["y=x^{-0.8}",.2,.35]),
      color=violet,explicit(x^-.4,x,0,3),label(["y=x^{-0.4}",.2,.15]),
      color=brown,point_type=7,points([[1,1]]))$
```



> **Exponential function** with base $a > 0$ is called
>
> $f\colon\ y = a^x,\ x \in R.$

- The most important is $f\colon\ y = \exp x = \mathrm{e}^x$ based on e (Euler's number).
- For $a = 1$, $f\colon\ y = 1^x = 1$ is constant (polynomial).
- For $a \in (0; 1)$ is f decreasing, for $a \in (1; \infty)$ is f increasing.
- The graph is called **exponential curve** and passes through the points $[0; 1]$ and $[1; a]$.

- The graphs of the functions $y = a^x$, $y = a^{-x}$ are symmetric along the axis y.



Functions $f\colon\ y = a^x$, $a > 0$ (left) a $f\colon\ y = \log_a x$, $a > 0$, $a \neq 1$ (right)

Exponential function `exp(x)=%e^x` and logarithmic function`log(x)` (natural logarithm) are based on e. If we want to calculate and different logarithm, e.g. $\log_2 x$, we must use construction $\log_2 x = \ln x / \ln 2$.

```
(%i1) exp(x)+%e^x;exp(1);
(%o1) 2*%e^x
(%o2) %e
(%i5) log(x);log(2);log(%e);
(%o3) log(x)
(%o4) log(2)
(%o5) 1
(%i7) log_2(x):=log(x)/log(2);log_2(2);
(%o6) log_2(x) := log(x)/log(2)
(%o7) 1
```

Logarithmic function with base $a > 0$, $a \neq 1$ is called

$$f\colon\ y = \log_a x,\ x \in (0; \infty).$$

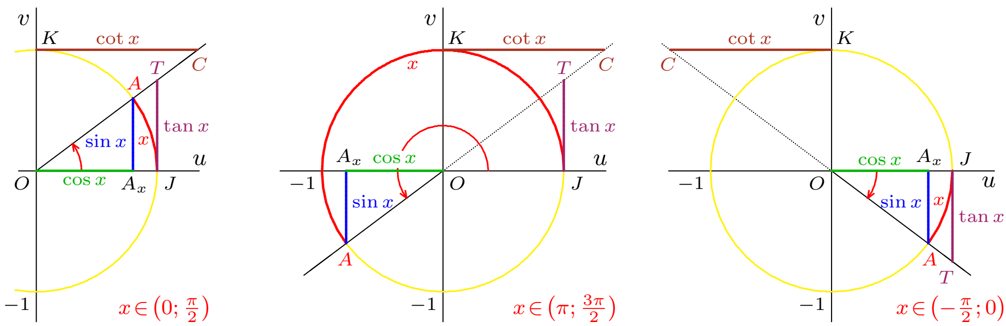
- f is inverse to the exponential function $y = a^x$, $x \in R$ with the same base $a > 0$, $a \neq 1$.
- For $x \in (0; \infty)$, $a > 0$, $a \neq 1$ holds $f\colon\ y = \log_a x \Leftrightarrow x = a^y$.
- $a > 0,\ a \neq 1. \quad \Rightarrow \begin{cases} x = a^{\log_a x} & \text{for } x > 0, \\ x = \log_a a^x & \text{for } x \in R. \end{cases}$
- For $a \in (0; 1)$ is f decreasing, for $a \in (1; \infty)$ is f increasing.
- The graph is called the **logarithmic curve** and passes through the points $[1; 0]$ and $[a; 1]$.
- Graphs of functions $y = \log_a x$, $y = \log_{and^{-1}} x$ are symmetric along the axis x.

The number $\log_a x$ is called the **logarithm of the number x based on a**.

- $a = 10$. $\Rightarrow$ **Decimal logarithm** numbers x, labeling $\log x$.
- $a = \mathrm{e}$. $\Rightarrow$ **Natural logarithm** of the number x, labeling $\ln x$.

Trigonometric (trigonometric) functions are:

- **Sine** $y = \sin x = |AA_x|$, $x \in R$.
- **Cosine** $y = \cos x = |OA_x|$, $x \in R$.
- **tangent** $y = \tan x = \frac{\sin x}{\cos x} = |TJ|$, $x \in R - \{\frac{\pi}{2} + k\pi;\ k \in Z\}$.
- **cotangent** $y = \cot x = \frac{\cos x}{\sin x} = |CK|$, $x \in R - \{k\pi;\ k \in Z\}$.



Defining functions $\sin x$, $\cos x$, $\tan x$, $\cot x$

Trigonometric functions are defined on and circle centered at the origin of and coordinate system with and radius of 1.

- The number π is called **Ludolf's**. Its value is approximately 3.141592654.
- A circle with radius $r = 1$ has a perimeter 2π.

$f\colon\ y = \sin x,\ D(f) = R,\ H(f) = \langle -1; 1\rangle$.

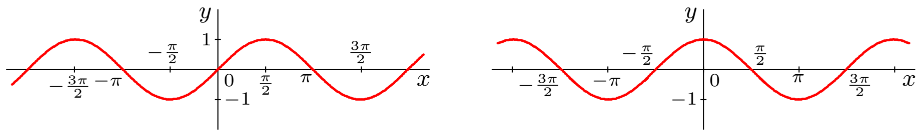
- f is odd, f is periodic with a primitive period of 2π.
- The graph f is called **sine wave**, zero points are $k\pi$, $k \in Z$.

$f\colon\ y = \cos x,\ D(f) = R,\ H(f) = \langle -1; 1\rangle$.

- f is even, f is periodic with a primitive period 2π.
- The graph f is called **cosine wave**, zero points are $\frac{\pi}{2} + k\pi$, $k \in Z$.

In the Maxima program, trigonometric functions have the form `sin(x)`, `cos(x)`, `tan(x)`, `cot(x)`. Arguments of trigonometric functions must be entered in radians. If we want to use degrees, we must first make a conversion to radians.

```
(%i3) tangrad(x):=tan(x/180*%pi);tangrad(22.5);ratsimp(tangrad(22.5));
```



Functions $f\colon\ y=\sin x$ (left) a $f\colon\ y=\cos x$ (right)

(%o1) $\mathrm{tangrad}(x) := \tan\left(\frac{x}{180}\pi\right)$
(%o2) $\tan(0.125\pi)$
rat : replaced 0.125 *by* 1/8 = 0.125
(%o3) $\tan\left(\frac{\pi}{8}\right)$

We can use commands to simplify work with trigonometric functions `trigsimp`, `trigrat`, `trigexpand`, `trigreduce` and packages `atrig1`, `ntrig` a `spangl`, which contain additional support for working with trigonometric functions. Packages must be loaded into the system using the command `load`.

(%i1) `tan(%pi/4);tan(%pi/6);tan(%pi/8);`

(%o3) $1 \quad \frac{1}{\sqrt{3}} \quad \tan\left(\frac{\pi}{8}\right)$

(%i4) `ratsimp(tan(%pi/8));`

(%o4) $\tan\left(\frac{\pi}{8}\right)$

(%i5) `trigsimp(tan(%pi/8));`

(%o5) $\frac{\sin\left(\frac{\pi}{8}\right)}{\cos\left(\frac{\pi}{8}\right)}$

(%i6) `load(spangl);`

(%o6) `"../share/trigonometry/spangl.mac"`

(%i7) `tan(%pi/8);`

(%o7) $\sqrt{2}-1$

Sum and difference formulas for sine a cosine

$x, y \in R. \Rightarrow$
- $\sin(x \pm y) = \sin x \cdot \cos y \pm \cos x \cdot \sin y.$
- $\cos(x \pm y) = \cos x \cdot \cos y \mp \sin x \cdot \sin y.$

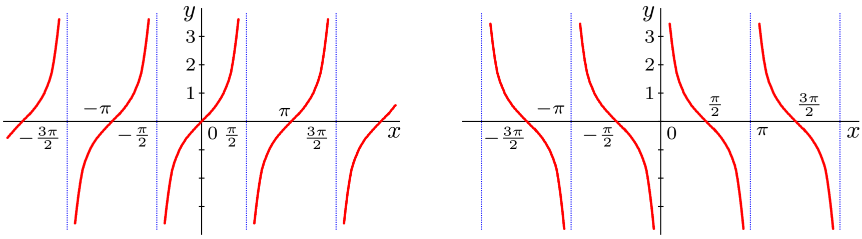
$x \in R. \Rightarrow$
- $\sin 2x = 2\sin x \cdot \cos x.$
- $\cos 2x = \cos^2 x - \sin^2 x.$
- $\sin^2 x = \frac{1-\cos 2x}{2}.$
- $\cos^2 x = \frac{1+\cos 2x}{2}.$

$f\colon\ y=\tan x,\ D(f)=R-\left\{\frac{\pi}{2}+k\pi;\ k\in Z\right\},\ H(f)=R.$

- f is odd, f is periodic with a primitive period π.
- Zero points are $k\pi,\ k\in Z$.

$f\colon\ y=\cot x,\ D(f)=R-\{k\pi;\ k\in Z\},\ H(f)=R.$

- f is odd, f is periodic with a primitive period π.
- Zero points are $\frac{\pi}{2}+k\pi,\ k\in Z$.



Functions $f\colon\ y=\tan x$ (left) a $f\colon\ y=\cot x$ (right)

Cyclometric (antitrigonometric) functions are inverse to trigonometric functions:

- **Arcsine** $y=\arcsin x\colon\ \langle -1;1\rangle\to\left\langle\frac{\pi}{2};\frac{\pi}{2}\right\rangle.$
- **Arccosine** $y=\arccos x\colon\ \langle -1;1\rangle\to\langle 0;\pi\rangle.$
- **Arctangent** $y=\operatorname{arctg} x\colon\ R\to\left(-\frac{\pi}{2};\frac{\pi}{2}\right).$
- **Arccotangent** $y=\operatorname{arccotg} x\colon\ R\to(0;\pi).$

There are no inverse functions to trigonometric functions because they are not injective. They need to be narrowed accordingly.

$y=\arcsin x,\ D(f)=\langle -1;1\rangle,\ H(f)=\left\langle\frac{\pi}{2};\frac{\pi}{2}\right\rangle.$

- f is increasing, f is odd.

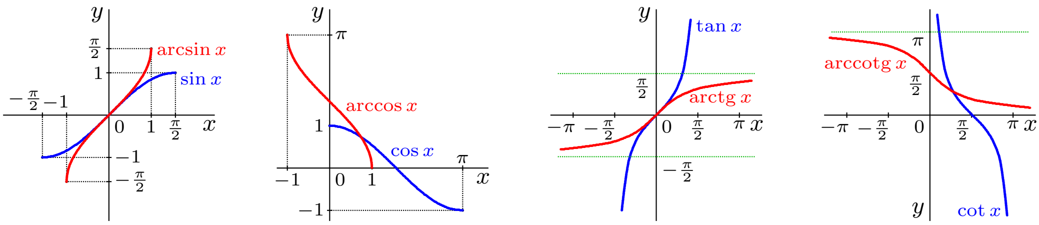
$y=\arccos x,\ D(f)=\langle -1;1\rangle,\ H(f)=\langle 0;\pi\rangle.$

- f is decreasing.

The inverse functions to trigonometric functions have the form in the Maxima program `asin(x)`, `acos(x)`, `atan(x)`, `acot(x)`.

At this point we can mention the function `atan2(x,y)` defined by $\operatorname{arctg}\frac{x}{y}$.

```
(%i2)  asin(1);acos(1);
(%o2)  π/2  0
(%i4)  atan2(2,4);atan(1/2);
(%o4)  atan(1/2)  atan(1/2)
```



Functions $y = \arcsin x$, $y = \arccos x$, $y = \operatorname{arctg} x$, $y = \operatorname{arccotg} x$

Sum formulas for Cyclometric Functions

$x \in \langle -1; 1 \rangle.$ $\Rightarrow$ • $\arcsin x + \arccos x = \frac{\pi}{2}.$

$x \in R.$ $\Rightarrow$ • $\operatorname{arctg} x + \operatorname{arccotg} x = \frac{\pi}{2}.$

$y = \operatorname{arctg} x$, $D(f) = R$, $H(f) = \left(-\frac{\pi}{2}; \frac{\pi}{2}\right).$

- f is increasing, f is odd.

$y = \operatorname{arccotg} x$, $D(f) = R$, $H(f) = (0; \pi).$

- f is decreasing.

Hyperbolic functions are:

- **Hyperbolic sine** $y = \sinh x = \frac{e^x - e^{-x}}{2} = \frac{e^{2x} - 1}{2\,e^x}$: $R \to R.$
- **Hyperbolic cosine** $y = \cosh x = \frac{e^x + e^{-x}}{2} = \frac{e^{2x} + 1}{2\,e^x}$: $R \to \langle 1; \infty).$
- **Hyperbolic tangent**

$$y = \coth x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}: \quad R \to (-1; 1).$$

- **Hyperbolic cotangent**

$$y = \coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}: \quad (R - \{0\}) \to (R - \langle -1; 1 \rangle).$$

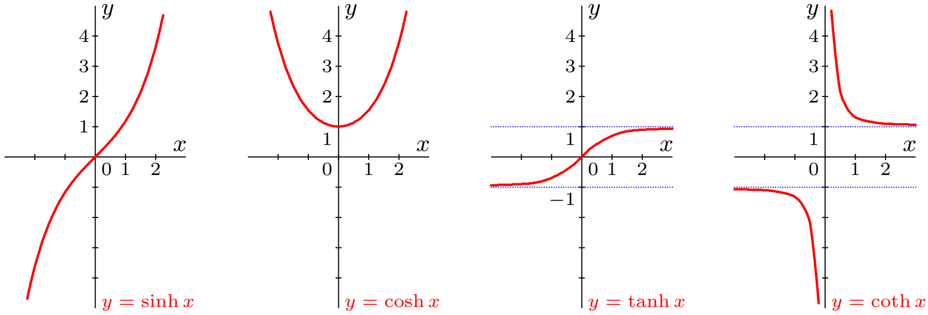
Hyperbolic functions have similar properties as trigonometric functions, therefore they have similar names.

$f\colon\ y = \sinh x = \frac{e^x - e^{-x}}{2}$, $D(f) = R$, $H(f) = R.$

- f is odd, f is increasing.

$f\colon\ y = \cosh x = \frac{e^x + e^{-x}}{2}$, $D(f) = R$, $H(f) = \langle 1; \infty).$

- f is even, f is decreasing to $(-\infty; 0)$, f is increasing to $(0; \infty).$



Hyperbolic functions $\sinh x$, $\cosh x$, $\tanh x$ and $\coth x$

Sum and difference formulas for hyperbolic sine a hyperbolic cosine

$x, y \in R.$ $\Rightarrow$
- $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y.$
- $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y.$

$x \in R.$ $\Rightarrow$
- $\sinh 2x = 2 \sinh x \cosh x,$
- $\cosh 2x = \cosh^2 x + \sinh^2 x.$
- $\sinh^2 x = \frac{\cosh 2x - 1}{2},$
- $\cosh^2 x = \frac{\cosh 2x + 1}{2}.$
- $\sinh x \pm \cosh x = \pm e^{\pm x},$
- $\cosh^2 x - \sinh^2 x = 1.$

Moivre's formula

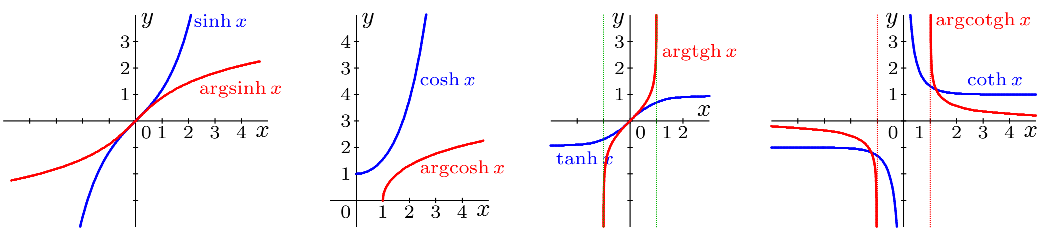
$x \in R,\ n \in N.$ $\Rightarrow$
- $(\cosh x \pm \sinh x)^n = \cosh nx \pm \sinh nx.$

$f\colon\ y = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}},\ D(f) = R,\ H(f) = (-1; 1).$

- f is odd, f is increasing.

$f\colon\ y = \coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}},\ D(f) = R - \{0\},\ H(f) = (-\infty; -1) \cup (1; \infty).$

- f is odd, f is decreasing on $(-\infty; 0)$, f is decreasing on $(0; \infty)$.



Functions $y = \operatorname{argsinh} x$, $y = \operatorname{argcosh} x$, $y = \operatorname{argtgh} x$, $y = \operatorname{argcotgh} x$

Area functions are inverse to hyperbolic functions:

- **Inverse hyperbolic sine**
$$y = \operatorname{argsinh} x = \ln\left(x + \sqrt{x^2+1}\right)\colon\ R \to R.$$
- **Inverse hyperbolic cosine**
$$y = \operatorname{argcosh} x = \ln\left(x + \sqrt{x^2-1}\right)\colon\ \langle 1;\infty) \to \langle 0;\infty).$$
- **Inverse hyperbolic tangent**
$$y = \operatorname{argtgh} x = \tfrac{1}{2}\ln\tfrac{1+x}{1-x}\colon\quad (-1;1) \to R.$$
- **Inverse hyperbolic cotangent**
$$y = \operatorname{argcotgh} x = \tfrac{1}{2}\ln\tfrac{x+1}{x-1}\colon\ (R - \langle -1;1\rangle) \to (R - \{0\}).$$

$f\colon\ y = \operatorname{argsinh} x = \ln\left(x + \sqrt{x^2+1}\right)$, $D(f) = R$, $H(f) = R$.

- f is odd, f is increasing.

$f\colon\ y = \operatorname{argcosh} x = \ln\left(x + \sqrt{x^2-1}\right)$, $D(f) = \langle 1;\infty)$, $H(f) = \langle 0;\infty)$.

- f is increasing.

$f\colon\ y = \operatorname{argtgh} x = \frac{1}{2}\ln\frac{1+x}{1-x}$, $D(f) = (-1;1)$, $H(f) = R$.

- f is odd, f is increasing.

$f\colon\ y = \operatorname{argcotgh} x = \frac{1}{2}\ln\frac{x+1}{x-1}$, $D(f) = (-\infty;-1) \cup (1;\infty)$, $H(f) = R - \{0\}$.

- f is odd, f is decreasing to $(-\infty;-1)$, f is decreasing to $(1;\infty)$.

Hyperbolic functions are `sinh(x)`, `cosh(x)`, `tanh(x)`, `coth(x)` and their inverse hyperbolometric functions are `asinh(x)`, `acosh(x)`, `atanh(x)`, `acoth(x)`.

```
(%i4)  sinh(x);cosh(0);tanh(0);coth(1),numer;
(%o1)  sinh(x)
(%o2)  1
(%o3)  0
(%o4)  1.313035285499331
(%i8)  asinh(x);acosh(1);atanh(0);acoth(1.3),numer;
(%o5)  asinh(x)
(%o6)  0
(%o7)  0
(%o8)  1.01844096363052
```

Limit of a function

When investigating a function, it is necessary to characterize its local properties at different intervals a in the neighborhood various important points. The function f does not have to be defined at the point around which we investigate it.

$a \in R^* = R \cup \{\pm\infty\}$ is called **set point** $A \subset R$,
if for each neighborhood $O(a)$ there exists a point $x \in O(a)$ such that $x \in a$, $x \neq a$.

f **has a limit** $b \in R^*$ at point $a \in R^*$, labeling $\lim\limits_{x\to a} f(x) = b$ if:

- a is the accumulation point of the set $D(f)$.
- For all $\{x_n\}_{n=1}^{\infty} \subset D(f)$, $x_n \neq a$, $\{x_n\}_{n=1}^{\infty} \longrightarrow a$ holds $\{f(x_n)\}_{n=1}^{\infty} \longrightarrow b$.

The second condition can be written in the form:

- $x_n \in D(f)$, $x_n \neq a$, $\lim\limits_{n\to\infty} x_n = a. \;\Rightarrow\; \lim\limits_{n\to\infty} f(x_n) = b.$

This definition is called **Heine's**.

- $\lim\limits_{x\to a} f(x) = b.$
  - $a \in R^*$.
    - $a \in R. \Rightarrow$ **Limit at real point** a.
    - $a = \pm\infty. \Rightarrow$ **Limit at infinity**.
  - $b \in R^*$.
    - $b \in R. \Rightarrow$ **Real limit**.
    - $b = \pm\infty. \Rightarrow$ **Infinity limit**.

The limit $\lim\limits_{x\to a} f(x) = b$ can be characterized by the neighborhood $O(a)$ and $O(b)$.

$a, b \in R^*$, $\lim\limits_{x\to a} f(x) = b. \;\Leftrightarrow$

- a is the accumulation point of the set $D(f)$.
- For each neighborhood $O(b)$ there is a neighborhood $O(a)$ such that that for all $x \in O(a)$, $x \neq a$ $f(x) \in O(b)$.

The second condition can be written in the form:

- For each neighborhood $O(b)$ there exists a neighborhood $O(a)$ such that $f\big(O(a) - \{a\}\big) \subset O(b)$.

If we use the radius of neighborhoods, we can write the second condition in the form:

- For every $O_\varepsilon(b)$ there is $O_\delta(a)$ so that for all $x \in O_\delta(a)$, $x \neq a$ $f(x) \in O_\varepsilon(b)$.

Specially applies:

- $\lim\limits_{x\to a} f(x) = b$, $a, b \in R$.
  $\Leftrightarrow\; \forall\varepsilon > 0\; \exists\delta > 0\; \forall x \in D(f)\colon\; 0 < |x - a| < \delta. \;\Rightarrow\; |f(x) - b| < \varepsilon.$
- $\lim\limits_{x\to\pm\infty} f(x) = b$, $b \in R$.
  $\Leftrightarrow\; \forall\varepsilon > 0\; \exists\delta \in R\; \forall x \in D(f)\colon\; \delta < x$, resp. $x < \delta. \;\Rightarrow\; |f(x) - b| < \varepsilon.$
- $\lim\limits_{x\to a} f(x) = \pm\infty$, $a \in R$.
  $\Leftrightarrow\; \forall\varepsilon \in R\; \exists\delta > 0\; \forall x \in D(f)\colon\; 0 < |x - a| < \delta. \;\Rightarrow\; \varepsilon < f(x)$, resp. $f(x) < \varepsilon.$

- $\lim\limits_{x\to\pm\infty} f(x) = \pm\infty.$

  $\Leftrightarrow\ \forall\varepsilon\in R\ \exists\delta\in R\ \forall x\in D(f)$: $\delta < x$, resp. $x < \delta$. $\Rightarrow \varepsilon < f(x)$, resp. $f(x) < \varepsilon$.

$a \in R^*$, $\lim\limits_{x\to a} f(x) = b \in R$. $\Rightarrow$ • There is a neighborhood $O(a)$ in which f is bounded.

The following statements represent the basic properties of function limits.

$a\in R^*$ is an accumulation point $D(f)$ and $D(g)$.
$f(x) = g(x)$ for all $x\in O(a)$, $x\neq a$. $\Rightarrow$

- Exists $\lim\limits_{x\to a} f(x)$. $\Leftrightarrow$ Exists $\lim\limits_{x\to a} g(x)$.
- $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x)$, if limits exist.

$a\in R^*$ is an accumulation point $D(f)$ and $D(g)$.
$f(x) \le g(x)$ for all $x\in O(a)$, $x\neq a$. $\Rightarrow$

- $\lim\limits_{x\to a} f(x) \le \lim\limits_{x\to a} g(x)$, if limits exist.

- If we change the assumption to $f(x) < g(x)$ for all $x\in O(a)$, $x\neq a$.

  $\Rightarrow$ The statement $\lim\limits_{x\to a} f(x) \le \lim\limits_{x\to a} g(x)$ does not change.

$a\in R^*$ is an accumulation point $D(f)$, $D(g)$ and $D(h)$.
$h(x) \le f(x) \le g(x)$ for all $x\in O(a)$, $x\neq a$.
Existujú $\lim\limits_{x\to a} h(x) = \lim\limits_{x\to a} g(x) = b\in R^*$. $\Rightarrow$ • Exists $\lim\limits_{x\to a} f(x) =$ b.

$\lim\limits_{x\to\infty} \frac{\sin x}{x} = 0.$

```
(%i1) limit(sin(x)/x,x,inf);
(%o1) 0
```

∞ is the accumulation point of the function scope $y = \frac{\sin x}{x}$.

For all $x \in R$ holds $-1 \leq \sin x \leq 1$. $\Rightarrow$ For all $x > 0$ holds $-\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$.

$$\Rightarrow \; 0 = -\lim_{x\to\infty} \frac{1}{x} \leq \lim_{x\to\infty} \frac{\sin x}{x} \leq \lim_{x\to\infty} \frac{1}{x} = 0. \;\Rightarrow\; \lim_{x\to\infty} \frac{\sin x}{x} = 0.$$

(%i1) `f(x):=sin(x)/x$ for i:1 thru 10 do (x:100^i, print(x," ",ev(f(x),numer)))$`

100 -0.005063656411097588
10000 $-3.056143888882521 \cdot 10^{-5}$
1000000 $-3.499935021712929 \cdot 10^{-7}$
100000000 $9.31639027109726 \cdot 10^{-9}$
10000000000 $-4.875060250875107 \cdot 10^{-11}$
1000000000000 $-6.112387023768895 \cdot 10^{-13}$
100000000000000 $-2.094083074964523 \cdot 10^{-15}$
10000000000000000 $7.796880066069787 \cdot 10^{-17}$
1000000000000000000 $-9.929693207404051 \cdot 10^{-19}$
100000000000000000000 $-6.452512852657808 \cdot 10^{-21}$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

(%i1) `limit(sin(x)/x,x,0);`
(%o1) 1

Let $f(x) = \frac{\sin x}{x}$, $x \in D(f) = \left(-\frac{\pi}{2}; \frac{\pi}{2}\right) - \{0\}$, point 0 is an accumulation point $D(f)$.

For all $x \in D(f)$ holds:

$$\left.\begin{array}{l} 0 < x. \;\Rightarrow\; 0 < \sin x < x < \tan x. \;\Rightarrow\; \frac{\sin x}{\sin x} < \frac{x}{\sin x} < \frac{\frac{\sin x}{\cos x}}{\sin x} = \frac{1}{\cos x}. \\ x < 0. \;\Rightarrow\; \tan x < x < \sin x < 0. \;\Rightarrow\; \frac{1}{\cos x} = \frac{\frac{\sin x}{\cos x}}{\sin x} > \frac{x}{\sin x} > \frac{\sin x}{\sin x}. \end{array}\right\} \Rightarrow\; 1 < \frac{x}{\sin x} < \frac{1}{\cos x}.$$

$$\Rightarrow\; 1 = \lim_{x\to 0} 1 \leq \lim_{x\to 0} \frac{x}{\sin x} \leq \lim_{x\to 0} \frac{1}{\cos x} = 1. \;\Rightarrow\; \lim_{x\to 0} \frac{x}{\sin x} = 1.$$

(%i1)
```
f(x):=sin(x)/x$
for i:-1 thru-10  step-1 do (x:1/i, print(x," ",ev(f(x),numer)))$
print("Limit")$
for i:10 thru 1 step-1 do (x:1/i, print(x," ",ev(f(x),numer)))$
```

-1 0.8414709848078965
$-\frac{1}{2}$ 0.958851077208406
$-\frac{1}{3}$ 0.9815840903884566
$-\frac{1}{4}$ 0.9896158370180917
$-\frac{1}{5}$ 0.9933466539753061
$-\frac{1}{6}$ 0.9953767961604901
$-\frac{1}{7}$ 0.9966021085458455
$-\frac{1}{8}$ 0.9973978670818215
$-\frac{1}{9}$ 0.9979436565895768
$-\frac{1}{10}$ 0.9983341664682815
Limit

$\frac{1}{10}$	0.9983341664682815
$\frac{1}{9}$	0.9979436565895768
$\frac{1}{8}$	0.9973978670818215
$\frac{1}{7}$	0.9966021085458455
$\frac{1}{6}$	0.9953767961604901
$\frac{1}{5}$	0.9933466539753061
$\frac{1}{4}$	0.9896158370180917
$\frac{1}{3}$	0.9815840903884566
$\frac{1}{2}$	0.958851077208406
1	0.8414709848078965

Limit of function composition

$y=f(x)$, $y=g(x)$, $H(f)\subset D(g)$.
$a,b,c\in R^*$, $\lim\limits_{x\to a} f(x)=b$, $\lim\limits_{u\to b} g(u)=c$.
$\begin{cases} f(x)\neq b \text{ for all } x\in O(a)-\{a\}, \\ \text{resp. } g(b)=c. \end{cases}$

$\Rightarrow$ • $\lim\limits_{x\to a} g(f(x)) = \lim\limits_{u\to b} g(u) = c.$

- When calculating $\lim\limits_{x\to a} g(f)$ we put $u=f(x)$. $\Rightarrow$ **Substitution** $u=f(x)$.
- $\lim\limits_{x\to a} f(x)$, $x\to a$, $x=h+a$. $\Rightarrow$ • $\lim\limits_{x\to a} f(x) = \lim\limits_{h\to 0} f(h+a)$, $h\to 0$.

$a,b,c\in R^*$, $r\in R$, $\lim\limits_{x\to a} f(x)=b$, $\lim\limits_{x\to a} g(x)=c$. $\Rightarrow$ (If the terms make sense.)

- $\lim\limits_{x\to a} |f(x)| = \left|\lim\limits_{x\to a} f(x)\right| = |b|.$
- $\lim\limits_{x\to a} \big(f(x)\pm g(x)\big) = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x) = b\pm c.$
- $\lim\limits_{x\to a} r f(x) = r \lim\limits_{x\to a} f(x) = rb.$
- $\lim\limits_{x\to a} \big(f(x)\cdot g(x)\big) = \lim\limits_{x\to a} f(x)\cdot \lim\limits_{x\to a} g(x) = bc.$
- $\lim\limits_{x\to a} \frac{1}{g(x)} = \frac{1}{\lim\limits_{x\to a} g(x)} = \frac{1}{c}.$
- $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)} = \frac{b}{c}.$

If one of the terms does not make sense, it does not necessarily mean the absence of a limit. We have to calculate the limit in a different way.

$y=f(x)$, $x\in D(f)$, point $a\in R$, denote:

- $f^-(x) = f(x)|_{D(f)\cap(-\infty;a)} = f(x)|_{\{x\in D(f),\, x<a\}}$ restriction function to the left.
- $f^+(x) = f(x)|_{D(f)\cap(a;\infty)} = f(x)|_{\{x\in D(f),\, a<x\}}$ restriction function to the right.

Limit from left a **limit from right of function** f **at point** $a\in R$ are called limits:

- $\lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a} f^-(x) = \lim\limits_{x\to a} \big[f|_{D(f)\cap(-\infty;a)}(x)\big].$
- $\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a} \big[f|_{D(f)\cap(a;\infty)}(x)\big].$

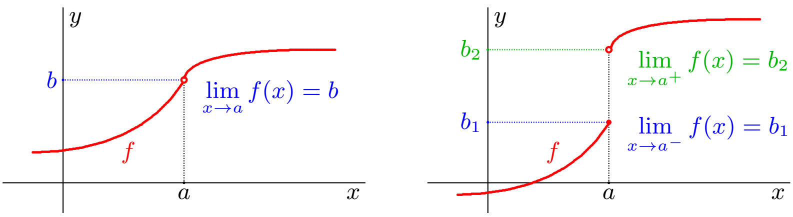
} **One-sided limits**.

$a \in R$, $b \in R^*$. $\Rightarrow$

- $\lim\limits_{x\to a} f(x) = b.$ $\Leftrightarrow$ • $\lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a^+} f(x) = b.$

- $\lim\limits_{x\to a} f(x)$ is called **two-sided limit**.

```
(%i3) limit(1/x,x,0,minus);
      limit(1/x,x,0);
      limit(1/x,x,0,plus);
(%o1) -∞
(%o2) infinity /* Complex inf */
(%o3) ∞
```



Two-sided limit and one-sided limits

Important limits.

- $\lim\limits_{x\to 0} \frac{\sin x}{x} = \lim\limits_{x\to 0} \frac{x}{\sin x} = 1.$
- $\lim\limits_{x\to 0} \frac{\arcsin x}{x} = \lim\limits_{x\to 0} \frac{x}{\arcsin x} = 1.$
- $\lim\limits_{x\to\infty} a^{\frac{1}{x}} = \lim\limits_{x\to\infty} \sqrt[x]{a} = 1$ for $a > 0$.
- $\lim\limits_{x\to\infty} x^{\frac{1}{x}} = \lim\limits_{x\to\infty} \sqrt[x]{x} = 1.$
- $\lim\limits_{x\to\infty} \left(1 + \frac{b}{x}\right)^x = \mathrm{e}^b$ for $b \in R$.
- $\lim\limits_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = \mathrm{e}.$
- $\lim\limits_{x\to 0} \frac{a^x - 1}{x} = \ln a$ for $a > 0$.
- $\lim\limits_{x\to 0} \frac{\mathrm{e}^x - 1}{x} = \ln \mathrm{e} = 1.$
- $\lim\limits_{x\to\infty} x\left(a^{\frac{1}{x}} - 1\right) = \ln a$ for $a > 0$.
- $\lim\limits_{x\to\infty} x\left(\mathrm{e}^{\frac{1}{x}} - 1\right) = 1.$
- $\lim\limits_{x\to\infty} \frac{x^q}{a^x} = \begin{cases} \infty \text{ for } a \in (0;1),\ q \in R, \\ 0 \text{ for } a \in (1;\infty),\ q \in R. \end{cases}$
- $\lim\limits_{x\to\infty} \frac{a^x}{x^q} = \begin{cases} 0 \text{ for } a \in (0;1),\ q \in R, \\ \infty \text{ for } a \in (1;\infty),\ q \in R. \end{cases}$

```
(%i2) limit(x*(%e^(1/x)-1),x,0); limit(x*(%e^(1/x)-1),x,inf);
(%o1) und        /* undefined */
(%o2) 1
```

- $\lim\limits_{x\to 2}\frac{x-2}{x^2-3x+2}=\lim\limits_{x\to 2}\frac{x-2}{(x-2)(x-1)}=\lim\limits_{x\to 2}\frac{1}{x-1}=\frac{1}{2-1}=1.$

- $\lim\limits_{x\to 0}\frac{3x+2x^{-1}}{x+4x^{-1}}=\lim\limits_{x\to 0}\frac{3x+2x^{-1}}{x+4x^{-1}}\cdot\frac{x}{x}=\lim\limits_{x\to 0}\frac{3x^2+2}{x^2+4}=\frac{3\cdot 0+2}{0+4}=\frac{1}{2}.$

- $\lim\limits_{x\to 2}\frac{x^2-3x+2}{x^2-2x}=\lim\limits_{x\to 2}\frac{(x-1)(x-2)}{x(x-2)}=\lim\limits_{x\to 2}\frac{x-1}{x}=\frac{2-1}{2}=\frac{1}{2}.$

```
(%i3) limit((x-2)/(x^2-3*x+2),x,2);
      limit((3*x+2*1/x)/(x+4*1/x),x,0);
      limit((x^2-3*x+2)/(x^2-2*x),x,2);
```

(%o1) 1

(%o2) $\frac{1}{2}$

(%o3) $\frac{1}{2}$

- $\lim\limits_{x\to 0}\frac{x}{\sqrt{1+x}-\sqrt{1-x}}=\lim\limits_{x\to 0}\frac{x\cdot(\sqrt{1+x}+\sqrt{1-x})}{(\sqrt{1+x}-\sqrt{1-x})\cdot(\sqrt{1+x}+\sqrt{1-x})}=\lim\limits_{x\to 0}\frac{x\cdot(\sqrt{1+x}+\sqrt{1-x})}{(1+x)-(1-x)}$

  $=\lim\limits_{x\to 0}\frac{x\cdot(\sqrt{1+x}+\sqrt{1-x})}{2x}=\lim\limits_{x\to 0}\frac{\sqrt{1+x}+\sqrt{1-x}}{2}=\frac{\sqrt{1+0}+\sqrt{1-0}}{2}=\frac{1+1}{2}=1.$

- $\lim\limits_{x\to 0}\frac{1-\sqrt{1-x}}{x}=\lim\limits_{x\to 0}\frac{(1-\sqrt{1-x})\cdot(1+\sqrt{1-x})}{x\cdot(1+\sqrt{1-x})}=\lim\limits_{x\to 0}\frac{1-(1-x)}{x+\sqrt{x^2-x^3}}=\lim\limits_{x\to 0}\frac{x}{x+\sqrt{x^2-x^3}}$

  $=\lim\limits_{x\to 0}\frac{x}{x+x\cdot\sqrt{1-x}}=\lim\limits_{x\to 0}\frac{1}{1+\sqrt{1-x}}=\frac{1}{1+\sqrt{1-0}}=\frac{1}{2}.$

- $\lim\limits_{x\to\infty}\frac{\sqrt{x^2-1}+\sqrt{x^2+1}}{x}=\lim\limits_{x\to\infty}\left(\sqrt{\frac{x^2-1}{x^2}}+\sqrt{\frac{x^2+1}{x^2}}\right)=\lim\limits_{x\to\infty}\left(\sqrt{1-\frac{1}{x^2}}+\sqrt{1+\frac{1}{x^2}}\right)$

  $=\sqrt{1-\frac{1}{\infty}}+\sqrt{1+\frac{1}{\infty}}=\sqrt{1-0}+\sqrt{1+0}=1+1=2.$

```
(%i3) limit(x/(sqrt(1+x)-sqrt(1-x)),x,0);
      limit((1-sqrt(1-x))/x,x,0);
      limit((\sqrt(x^2-1)+sqrt(x^2+1))/x,x,inf);
```

(%o1) 1

(%o2) $\frac{1}{2}$

(%o3) 2

- $\lim\limits_{x\to 1} \frac{\sqrt[3]{x}-1}{\sqrt[4]{x}-1} = \left[\begin{matrix}\text{Subst.} & x = z^{12} \\ x\to 1, & z \to 1\end{matrix}\right] = \lim\limits_{z\to 1} \frac{\sqrt[3]{z^{12}}-1}{\sqrt[4]{z^{12}}-1} = \lim\limits_{z\to 1} \frac{z^4-1}{z^3-1}$
$= \lim\limits_{z\to 1} \frac{(z-1)(z^3+z^2+z+1)}{(z-1)(z^2+z+1)} = \lim\limits_{z\to 1} \frac{z^3+z^2+z+1}{z^2+z+1} = \frac{1+1+1+1}{1+1+1} = \frac{4}{3}.$

- $\lim\limits_{x\to\infty} \left(\frac{5x^2}{x^2-1} + 2^{\frac{1}{x}}\right) = \lim\limits_{x\to\infty} \frac{5}{1-x^{-2}} + \lim\limits_{x\to\infty} 2^{\frac{1}{x}} = \frac{5}{1-\infty^{-2}} + 2^0 = \frac{5}{1-0} + 1 = 6.$

- $\lim\limits_{x\to 0+} x^{\frac{a}{\ln x}} = \lim\limits_{x\to 0+} e^{\ln x^{\frac{a}{\ln x}}} = \lim\limits_{x\to 0+} e^{\frac{a}{\ln x}\cdot \ln x} = \lim\limits_{x\to 0+} e^a = e^a$ for $a \in R$.

```
(%i3)  limit((x^(1/3)-1)/(x^(1/4)-1),x,1);
       limit(5*x^2/(x^2-1)+2^(1/x),x,inf);
       limit(x^(a/log(x)),x,0,plus);
```
(%o1) $\frac{4}{3}$
(%o2) 6
(%o3) e^a

If we use the substitution $x = z^{12}$, we can simplify the first limit.

```
(%i2)  f(x):=(x^(1/3)-1)/(x^(1/4)-1)$ g(z):=subst(z^12,x,f(x))$

       'limit(g(z),z,1); limit(g(z),z,1);
```
(%o1) $\lim\limits_{z\to 1} \frac{z^4-1}{z^2\cdot|z|-1}$ `/* z is positive, z=|z| */`

(%o2) $\frac{4}{3}$

In the last example, we calculated the limit of the expression 0^0 – so called indefinite form.

Indefinite form (calculated by limits) include:

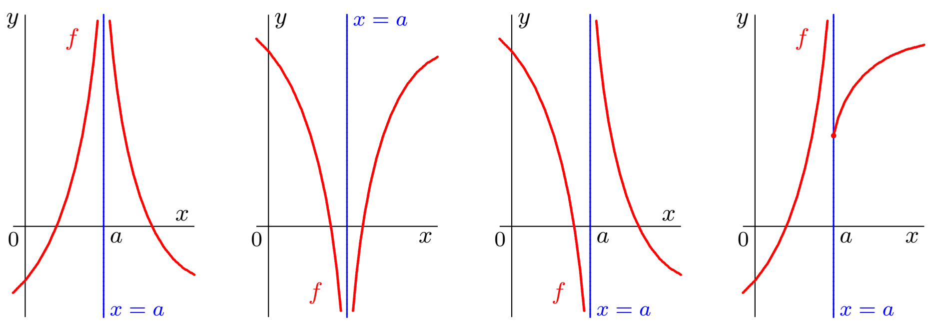
- $\infty - \infty$, - $\pm\infty \cdot 0$, - $\frac{0}{0}$, - $\frac{1}{0}$, - $\frac{\pm\infty}{0}$, - $\frac{\pm\infty}{\pm\infty}$, - 0^0, - $0^{\pm\infty}$, - $1^{\pm\infty}$, - $(\pm\infty)^0$.

- $\lim\limits_{x\to\infty} x\big(\ln(x+2)-\ln x\big) = \lim\limits_{x\to\infty} x\cdot\ln\frac{x+2}{x} = \lim\limits_{x\to\infty} \ln(1+\frac{2}{x})^x = \ln e^2 = 2.$

- $\lim\limits_{x\to 0} \frac{x}{\ln(1+tx)} = \begin{bmatrix}\text{Subst.} & z=tx\\ x\to 0, & z\to 0\end{bmatrix} = \lim\limits_{z\to 0} \frac{\frac{z}{t}}{\ln(1+z)} = \frac{1}{t}\cdot\lim\limits_{z\to 0}\frac{1}{\frac{1}{z}\cdot\ln(1+z)}$
  $= \frac{1}{t}\cdot\lim\limits_{z\to 0}\frac{1}{\ln(1+z)^{\frac{1}{z}}} = \frac{1}{t}\cdot\frac{1}{\ln e} = \frac{1}{t}\cdot\frac{1}{1} = \frac{1}{t}$ for $t\in R,\ t\neq 0.$

- $\lim\limits_{x\to\infty}\left(\frac{3x-2}{3x+1}\right)^x = \lim\limits_{x\to\infty}\left(\frac{3x+1-3}{3x+1}\right)^{\frac{3x+1-1}{3}} = \begin{bmatrix}\text{Subst.}\ z=3x+1\\ x\to\infty, \quad z\to\infty\end{bmatrix} = \lim\limits_{z\to\infty}\left(\frac{z-3}{z}\right)^{\frac{z-1}{3}}$
  $= \lim\limits_{z\to\infty}\left[\left(1-\frac{3}{z}\right)^z\right]^{\frac{z-1}{3z}} = \left[e^{-3}\right]^{\frac{1}{3}} = e^{-1} = \frac{1}{e}.$

```
(%i3) limit(x*(log(x+2)-log(x)),x,inf);
      limit(x/log(1+t*x),x,0);
      limit(((3*x-2)/(3*x+1))^x,x,inf);
(%o1) 2
(%o2) 1/t
(%o3) e^-1
```



Examples of asymptotes without a directive

Asymptotic Properties

When investigating the function f, it is important to examine its properties in improper points:

- For $x\to\pm\infty$.
- In the neighborhood $O(a)$ point $a\in R$ to which it applies $\lim\limits_{x\to a^-} f(x) = \pm\infty$ alebo $\lim\limits_{x\to a^+} f(x) = \pm\infty$.

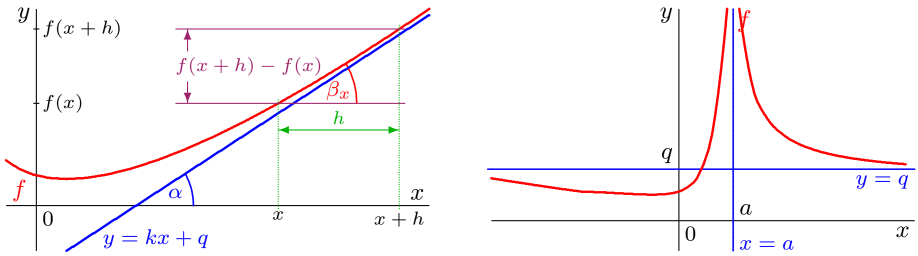
$y = f(x),\ x\in D(f),\ a\in R.$

- The line $x = a$ is called **asymptote without direction (vertical asymptote) of**

the graph f, if $\lim_{x\to a^-} f(x) = \pm\infty$ or $\lim_{x\to a^+} f(x) = \pm\infty$ (At least one of the limits is improper).

- The line $y = kx + q$ is called **asymptote with directive (oblique asymptote) of graph** f, if $\lim_{x\to -\infty} [f(x) - (kx+q)] = 0$ or $\lim_{x\to\infty} [f(x) - (kx+q)] = 0$.

  If $k = 0$ (line direction), the asymptote is called **horizontal**.



Examples of asymptotes

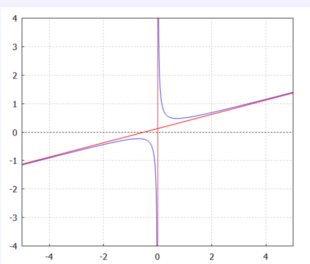
> The line $y = kx + q$ is an asymptote with the direction of the function $y = f(x)$, $x \in D(f)$.
>
> $\Leftrightarrow$ • Exist realn limits $\lim_{x\to\pm\infty} \frac{f(x)}{x} = k \in R$, $\lim_{x\to\pm\infty} [f(x) - kx] = q \in R$.

```
(%i10) f(x):=(2*x^2+x+1)/(8*x); km:limit(f(x)/x,x,minf)$ kp:limit(f(x)/x,x,inf)$
       qm:limit(f(x)-km*x,x,minf)$ qp:limit(f(x)-kp*x,x,inf)$
       dm(x):=km*x+qm$ dp(x):=kp*x+qp$ dm(x);dp(x);
       draw2d(grid=true,xaxis=true,yaxis=true,xrange=[-5,5],yrange=[-4,4],
       color=blue,explicit(f(x),x,-8,0),explicit(f(x),x,0,8),
       color=red,parametric(0,t,t,-5,5),
       explicit(dm(x),x,-8,8),explicit(dp(x),x,-8,8))$
```

(%o1) $f(x) := \frac{2x^2+x+1}{8x}$

(%o8) $\frac{x}{4} + \frac{1}{8}$

(%o9) $\frac{x}{4} + \frac{1}{8}$



- $y = kx + q$ is an asymptote with a directive. $\Rightarrow \lim\limits_{x\to\infty} [f(x) - (kx + q)] = 0.$

  $\lim\limits_{x\to\infty} \frac{f(x)-(kx+q)}{x} = \lim\limits_{x\to\infty} \left(\frac{f(x)}{x} - k - \frac{q}{x}\right) = 0.$ $\Rightarrow$ • $\lim\limits_{x\to\infty} \frac{f(x)}{x} = k.$

  $\lim\limits_{x\to\infty} [f(x) - (kx + q)] = \lim\limits_{x\to\infty} [(f(x) - kx) - q] = 0$ $\Rightarrow$ • $\lim\limits_{x\to\infty} [f(x) - kx] = q.$

- $\lim\limits_{x\to\pm\infty} \frac{f(x)}{x} = k \in R,\ \lim\limits_{x\to\pm\infty} [f(x) - kx] = q \in R.$

  $\Rightarrow$ • $\lim\limits_{x\to\infty} [f(x) - (kx + q)] = \lim\limits_{x\to\infty} [f(x) - kx] - \lim\limits_{x\to\infty} q = q - q = 0.$

Continuity of function

Closely related to the notion of the limit of the function f at the point a is the notion of the continuity of f at the point a. Connectivity is also a local issue in the neighborhood $O(a)$.

> **f is continuous at point $a \in D(f)$** if:
> - For all $\{x_n\}_{n=1}^{\infty} \subset D(f),\ x_n \neq a,\ \{x_n\}_{n=1}^{\infty} \longrightarrow a$ holds $\{f(x_n)\}_{n=1}^{\infty} \longrightarrow f(a).$

We can write the condition in the form:

- $x_n \in D(f),\ \lim\limits_{n\to\infty} x_n = a.$ $\Rightarrow$ $\lim\limits_{n\to\infty} f(x_n) = f(a).$

This definition is called **Heine's**.

The point $a \in D(f)$ can only be bulk or isolated:

- $a \in D(f)$ is an isolated point. $\Rightarrow$ There is only $\{x_n\}_{n=1}^{\infty} = \{a\}_{n=1}^{\infty} \longrightarrow to.$

  $\Rightarrow \lim\limits_{n\to\infty} f(x_n) = \lim\limits_{n\to\infty} f(a) = f(a).$

> $a \in D(f)$ is an isolated point. $\Rightarrow$ • f is continuous at a.

- $a \in D(f)$ is an accumulation point. $\Rightarrow$ The definition is identical to the definition of the f limit at a.

> $a \in D(f)$ is an accumulation point $D(f)$. $\Rightarrow$
> - f is continuous at a. $\Leftrightarrow$ • $\lim\limits_{x\to a} f(x) = f(a).$

The continuity of the function at the point $a \in D(f)$ can be characterized by the neighborhood $O(a)$ and $O(f(a))$.

f is continuous at point (f). $\Leftrightarrow$

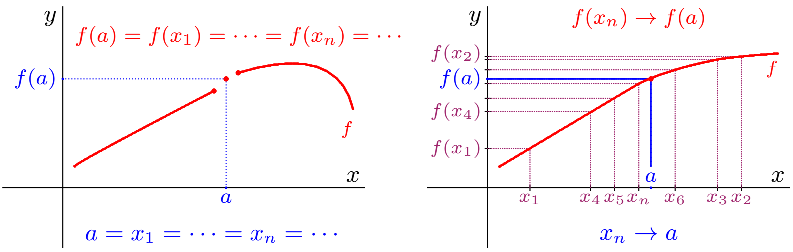
- For each neighborhood $O(f(a))$ there is a neighborhood $O(a)$ such that that $f(x) \in O(f(a))$ applies to all $x \in O(a)$.

We can write the condition in the form:

- For each neighborhood $O(f(a))$ there exists a neighborhood $O(a)$ such that $f\big(O(a)\big) \subset O(f(a))$.

If we use the radius of neighborhoods, then we can write:

- For every $O_\varepsilon(f(a))$ there is $O_\delta(a)$ so that for all $x \in O_\delta(a)$, $x \neq a$ $f(x) \in O_\varepsilon(b)$.
- $\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in D(f)$: $|x-a| < \delta. \ \Rightarrow\ |f(x)-f(a)| < \varepsilon$.



Continuity of f function at isolated point (left) and accumulation point (right)

The function f is called **discontinuous at point** $a \in D(f)$, if it is not continuous at point a:

- There is $\{x_n\}_{n=1}^{\infty} \subset D(f)$, $\{x_n\}_{n=1}^{\infty} \longrightarrow a$ so, that $\{f(x_n)\}_{n=1}^{\infty} \not\longrightarrow f(a)$, i.e. there is $\lim\limits_{n\to\infty} f(x_n) \neq f(a)$ or there is no $\lim\limits_{n\to\infty} f(x_n)$.

- f is continuous at $a \in D(f)$. $\Rightarrow$ • a is called **continuity point** f.
- f is discontinuous at point $a \in D(f)$. $\Rightarrow$ • a is called **discontinuity point** f.

f can only be discontinuous at the $D(f)$ accumulation point.
$\Rightarrow$ We will extend the concept of a discontinuity point to all mass points $D(f)$.

$y = f(x)$, $x \in D(f)$, a is an accumulation point $D(f)$.

- a is the point of **removable discontinuity of function** f,

  if $\lim\limits_{x\to a} f(x) \in R$ exists, $\lim\limits_{x\to a} f(x) \neq f(a)$.

- We remove the discontinuity in the point a if we define $f(a) = \lim\limits_{x\to a} f(x)$.

- a is the point of **irremovable discontinuity of the I. type of function** f,

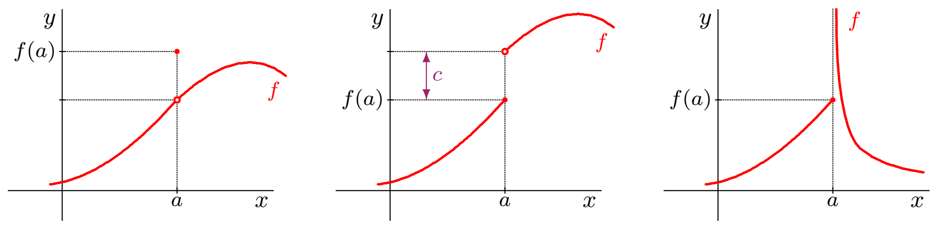
  if any $\lim\limits_{x\to a^-} f(x)\in R$, $\lim\limits_{x\to a^+} f(x)\in R$, $\lim\limits_{x\to a^-} f(x)\neq \lim\limits_{x\to a^+} f(x)$.

  - The number $c = \lim\limits_{x\to a^+} f(x) - \lim\limits_{x\to a^-} f(x)$ is called **jump of function** f **at point** a.

- a is the point of **unrecoverable discontinuity II. type of function** f,

  if at least one of the limits $\lim\limits_{x\to a^-} f(x)$, $\lim\limits_{x\to a^+} f(x)$ does not exist or is infinite.

  - If any of the limits are infinite, we are talking about **asymtotic discontinuity**.



Discontinuity removable, irreversible type I a irreversible type II

f, g are continuous at point $a\in D(f)\cap D(g)$, $r\in R$. $\Rightarrow$

- $|f|$, $f\pm g$, rf, fg are continuous at a.
- $g(a)\neq 0$. $\Rightarrow$ $\frac{1}{f}$, $\frac{f}{g}$ are continuous at a.

Continuity of a function composition

f is continuous at $a\in D(f)$.
g is continuous at $b = f(a)\in D(g)$.
$H(f)\subset D(g)$.

$\Rightarrow$ • $F = g(f)$ is continuous at a.

$a\in D(f)\cap D(g)\cap D(h)$, $O(a)$ is neighborhood.
g, h are continuous at point a.
$h(a) = f(a) = g(a)$.
$h(x) \leq f(x) \leq g(x)$ for all $x\in O(a)$.

$\Rightarrow$ • f is continuous at a.

$y = f(x)$, $x\in D(f)$, point $a\in D(f)$, denote:

- $f_a^-(x) = f(x)|_{D(f)\cap(-\infty;a\rangle} = f(x)|_{\{x\in D(f),\, x\leq a\}}$ restriction function to the left.
- $f_a^+(x) = f(x)|_{D(f)\cap\langle a;\infty)} = f(x)|_{\{x\in D(f),\, a\leq x\}}$ restriction function to the right.

The function $y = f(x)$, $x\in D(f)$ is called **at point** $a\in D(f)$:

- **continuous left** if there is a continuous function f_a^- in a.
- **continuous right** if there is a continuous function f_a^+ in a.

} **One-sided. continuity**.

f is continuous at point $a\in D(f)$. $\Leftrightarrow$ • f is a continuous from left and continuous from right at a.

Local boundary

f is continuous at point $a\in D(f)$. $\Rightarrow$ • There is a neighborhood $O(a)$ in which f is bounded.

$y = f(x)$, $x\in D(f)$, set $A\subset D(f)$.

- f is called **continuous on set** A, if it is continuous at each point $a\in a$.

The connection f on the set $A\subset D(f)$ does not follow boundary of f to A.

- f: $y = \frac{1}{x}$, $x\in R$ is continuous on the interval $(0;1)$, but is not limited to $(0;1)$.

Cauchy on zero point

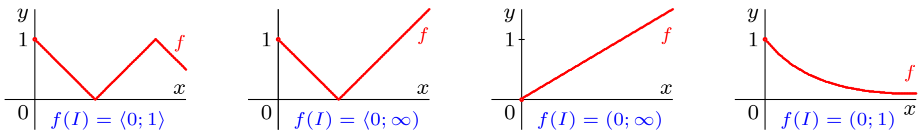
f is continuous on $\langle a;b\rangle$.
$f(a)\cdot f(b) < 0$.
} $\Rightarrow$ • There exists $c\in(a;b)$ such that $f(c) = 0$.

f is continuous on the interval $I\subset R$. $\Rightarrow$

- $f(I)$ is the interval.
- The inverse function f^{-1} (if any) is continuous on $f(I)$.

f is continuous on the interval $I\subset R$.

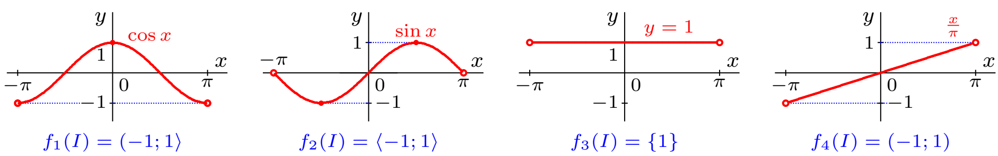
- I is a closed interval. $\Rightarrow$ • $f(I)$ is a closed interval.
- I is not a closed interval. $\Rightarrow$ • $f(I)$ can be an interval of different types.



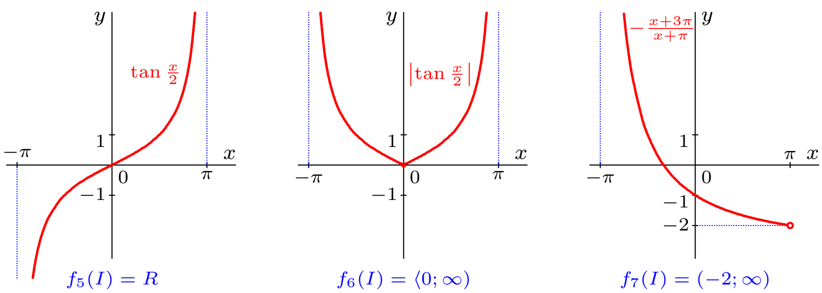
Display interval $I = (0;\,\infty)$ by continuous function f

A continuous function can display $I = (-\pi;\pi)$ at different intervals:

- $y = \cos x$: $(-\pi;\pi) \to (-1;1\rangle$.
- $y = \sin x$: $(-\pi;\pi) \to \langle -1;1\rangle$.
- $y = 1$: $(-\pi;\pi) \to \{1\}$.
- $y = \frac{x}{\pi}$: $(-\pi;\pi) \to (-1;1)$.
- $y = \tan\frac{x}{2}$: $(-\pi;\pi) \to R$.
- $y = \left|\tan\frac{x}{2}\right|$: $(-\pi;\pi) \to \langle 0;\infty)$.
- $y = -\frac{2x}{x+\pi} - 1$: $(-\pi;\pi) \to (0;\infty)$.
- $y = -\frac{2x}{x+\pi}$: $(-\pi;\pi) \to (1;\infty)$.



Display interval $I = (-\pi;\,\pi)$ by continuous functions



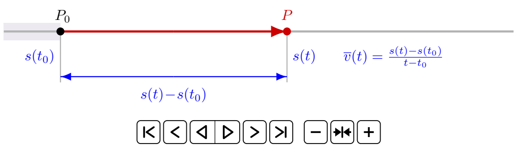
Display interval $I = (-\pi;\,\pi)$ by continuous functions

Derivative of a real function

Two problems led to the introduction of the derivative of a function (following examples).

A point moves along a line, its movement at time t is described by the function $y = s(t)$.

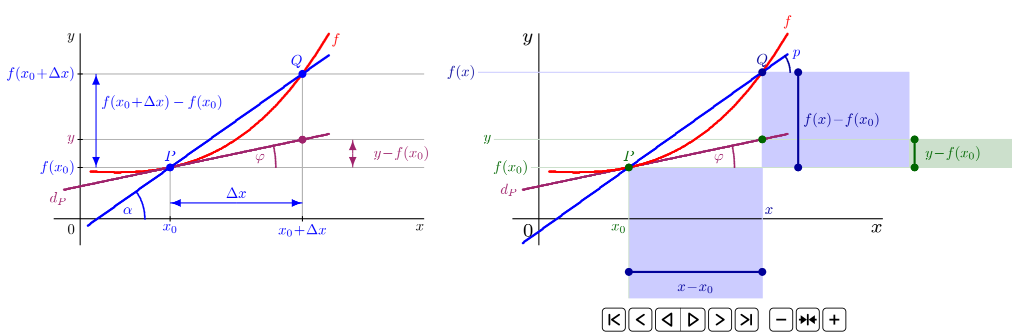
- At time t_0 it is at point P_0, at time t it is at point P.
- In the time interval $\langle t_0; t\rangle$ travels the path $s(t) - s(t_0)$.
  - $\Rightarrow$ • Average speed $\overline{v}(t) = \frac{s(t)-s(t_0)}{t-t_0}$.
- For $t \to t_0$ we get the instantaneous point velocity at time t_0.
  - $\Rightarrow$ • Instantaneous speed $v(t_0) = \lim\limits_{t\to t_0} \overline{v}(t) = \lim\limits_{t\to t_0} \frac{s(t)-s(t_0)}{t-t_0}$.



Speed Task

The function $y = f(x)$, $x \in D(f)$ is continuous.

- Points $P = [x_0; f(x_0)]$, $Q = [x_0+\Delta x; f(x_0+\Delta x)]$ lies on the graph f.
- The line PQ has the guideline $\tan\alpha = \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$.
- The tangent to f at point P has the form d_P: $y - f(x_0) = \tan\varphi \cdot \Delta x$, where $\tan\varphi = \frac{y-f(x_0)}{\Delta x}$ is its guideline.
- $Q \to P$. $\Rightarrow$ $PQ \to d_P$, $\Delta x \to 0$, $\alpha \to \varphi$, $f(x_0+\Delta x) \to f(x_0)$. $\Rightarrow$ $\tan\alpha \to \tan\varphi$.
  - $\Rightarrow$ • Directive of tangent $\tan\varphi = \lim\limits_{\alpha\to\varphi} \tan\alpha = \lim\limits_{\Delta x\to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$.



Tangent task

f has at point $x_0 \in D(f)$ **derivative**, labeling $f'(x_0)$, resp. $y'(x_0)$ if:

- Exists limit $f'(x_0) = \lim\limits_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \begin{bmatrix} \text{Subst. } h = x - x_0 \\ x \to x_0, \quad h \to 0 \end{bmatrix} = \lim\limits_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$.

- $f'(x_0) \in R$. $\Rightarrow$ The derivative $f'(x_0)$ is **finite**.
- $f'(x_0) \pm \infty$. $\Rightarrow$ The derivative $f'(x_0)$ is **infinite**.

The labeling introduced by G. W. Leibniz is often used:

- $f'(x_0) = \frac{\mathrm{d}f(x_0)}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x} f(x_0)$, resp. • $y'(x_0) = \frac{\mathrm{d}y(x_0)}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x} y(x_0)$.

$f'(x_0) \in R$ (is finite). $\Rightarrow$ • f is continuous at x_0.

- The continuity of the function f at the point x_0 does not guarantee the existence of $f'(x_0)$.

The function f: $y = |x|$ is continuous at $x_0 = 0$.

- Neexists $f'(0) = \lim\limits_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0} \frac{|x|-|0|}{x-0} = \lim\limits_{x\to 0} \frac{|x|}{x} = \begin{cases} \lim\limits_{x\to 0+} \frac{x}{x} = 1. \\ \lim\limits_{x\to 0+} \frac{-x}{x} = -1. \end{cases}$

- $f'(x_0) \in R$. $\Rightarrow$ $f'(x_0)$ is the direction of the tangent to the graph f at the point x_0. The tangent has the form $y = f(x_0) + f'(x_0)(x - x_0)$.
- $f'(x_0) = \pm\infty$, f is continuous at x_0.
  $\Rightarrow$ $x = x_0$ is a tangent (without direction) to the graph f at the point x_0.

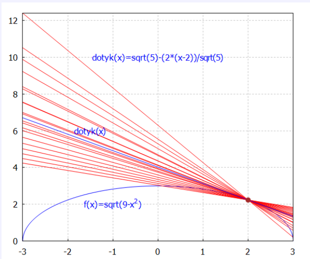
Determine the tangent to the semicircle $y = \sqrt{9 - x^2}$ at the point 2.

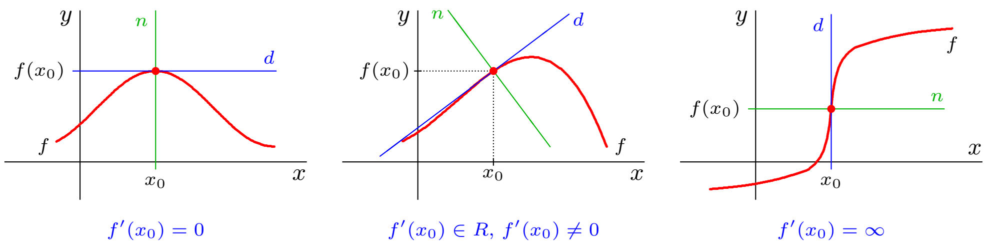
```
(%i8) f(x):=sqrt(9-x^2)$ pomer(a,b):=(f(b)-f(a))/(b-a)$
      sek(x,a,b):=pomer(a,b)*(x-a)+f(a)$
      Sek:makelist(explicit(sek(x,2,-.15+.25*i),x,-3,3),i,1,20)$
      f1(x):=diff(f(x),x,1)$ dotyk(x):=f(2)+subst(2,x,f1(x))*(x-2)$
      print("Secant y=dotyk(x)=",dotyk(x)," in point 2 have a blue color")$
      draw2d(grid=true,xaxis=true,color=blue,explicit(f(x),x,-3,3),
      color=red,Sek,color=blue,explicit(dotyk(x),x,-3,3),
      point_type=7,color=brown,points([[2,f(2)]]),
      color=blue,label(["f(x)=sqrt(9-x^2)",-1,2]),
```

```
label(["dotyk(x)",-1.5,6]),label([concat("dotyk(x)=",string(dotyk(x))),0,10]))$
```

Secant $y = dotyk(x) = \sqrt{5} - \frac{2(x-2)}{\sqrt{5}}$ *in point 2 have a blue color*

\





Tangent and normal to function f at x_0

We calculate a simplify the derivative of the function $f(x) = \ln(x + \sqrt{x^2+1})$.

(%i1) `f(x):=log(x+sqrt(x^2+1));`

(%o1) $f(x) := \log(x + \sqrt{x^2+1})$

(%i3) `f1(x):=diff(f(x),x);f1(x);`

(%o2) $f1(x) := \frac{\mathrm{d}}{\mathrm{d}x} f(x)$

(%o3) $\frac{\frac{x}{\sqrt{x^2+1}}+1}{\sqrt{x^2+1}+x}$

(%i4) `ratsimp(f1(x));`

(%o4) $\frac{\sqrt{x^2+1}+x}{x\sqrt{x^2+1}+x^2+1}$

We calculated the derivative $f'(x)$, but we could not simplify it appropriately. Use the command `subst`.

(%i5) `fp:subst(a,sqrt(x^2+1),f1(x));`

(fp) $\frac{\frac{x}{a}+1}{x+a}$

(%i6) `ratsimp(fp);`

(%o6) $\frac{1}{a}$

(%i7) `subst(sqrt(x^2+1),a,ratsimp(fp));`

(%o7) $\frac{1}{\sqrt{x^2+1}}$

f has at point $x_0 \in D(f)$ **derivative from the left** $f'_-(x_0)$, if:

- Exists $f'_-(x_0) = \lim\limits_{x \to x_0^-} \frac{f(x)-f(x_0)}{x-x_0} = \begin{bmatrix}\text{Subst. } h = x - x_0 \\ x \to x_0, \quad h \to 0\end{bmatrix} = \lim\limits_{h \to 0-} \frac{f(x_0+h)-f(x_0)}{h}$.

$f'_-(x_0) \in R$ (is finite). $\Rightarrow$ • f is a continuous from the left at the point x_0.

f has at point $x_0 \in D(f)$ **derivative from the right** $f'_+(x_0)$, if:

- Exists $f'_+(x_0) = \lim\limits_{x \to x_0^-} \frac{f(x)-f(x_0)}{x-x_0} = \begin{bmatrix}\text{Subst. } h = x - x_0 \\ x \to x_0, \quad h \to 0\end{bmatrix} = \lim\limits_{h \to 0+} \frac{f(x_0+h)-f(x_0)}{h}$.

$f'_+(x_0) \in R$ (is finite). $\Rightarrow$ • f is a continuous from right at the point x_0.

Derivatives $f'_-(x_0)$, $f'_+(x_0)$ are called **one-sided** a graphically represent directives **left**, resp. **right half-tangent to graph** f **at point** x_0. The derivative $f'(x_0)$ is called **two-sided**.

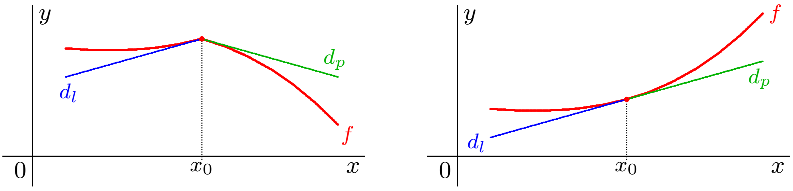
$f'_-(x_0), f'_+(x_0) \in R$ (they are finite, they do not have to be equal). $\Rightarrow$

• f is continuous at x_0.

There is $f'(x_0)$. $\Leftrightarrow$ • There are $f'_-(x_0)$, $f'_+(x_0)$ and $f'_-(x_0) = f'_+(x_0)$.

The following construction calculates a draws a tangent to the graph of the function f at point c.

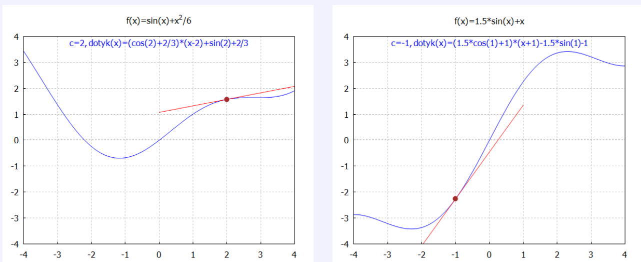
(%i6)
```
c:2$ f(x):=x^2/6+sin(x)$
f1(x):=diff(f(x),x,1)$ dotyk(x):=f(c)+subst(c,x,f1(x))*(x-c)$
print("Secant y=dotyk(x)=",dotyk(x)," in point",c)$
draw2d(grid=true,xaxis=true,xrange=[-4,4],yrange=[-4,4],
```



One-sided half-tangents

```
color=blue,explicit(f(x),x,-4,4),
color=red,explicit(dotyk(x),x,c-2,c+2),
point_type=7,color=brown,points([[c,f(c)]]),
color=blue,title=concat("f(x)=",string(f(x))),
label([concat("c=",string(c),", dotyk(x)=",string(dotyk(x))),0,3.75]))$
```

Secant $y = dotyk(x) = (\cos(2) + 2/3) * (x - 2) + \sin(2) + 2/3$ *in point* 2



$y = f(x),\ x \in D(f),\ A \subset \{x_0 \in D(f);\ f'(x_0) \text{ is finite}\},\ A \neq \emptyset.$

- The function g: $y = f'(x),\ x \in A$ is called **derivative of the function f on the set A**, labeling f', y', resp. $\frac{\mathrm{d}f}{\mathrm{d}x}$, $\frac{\mathrm{d}y}{\mathrm{d}x}$.

- The derivative f at the point $x_0 \in D(f)$ is $f'(x_0)$, i.e. number or $\pm\infty$.
- The derivative f on the set $A \subset D(f)$ is a function $y = f'(x),\ x \in a$.

f has a finite derivative f' on the set $A \subset D(f)$. $\Rightarrow$ • f is continuous on A.

$f\colon y = x^n,\ x\in R,\ n\in N,\ x_0\in D(f).$

- $f'(x_0) = \lim\limits_{x\to x_0} \frac{x^n - x_0^n}{x-x_0} = \lim\limits_{x\to x_0} \frac{(x^{n-1}+x^{n-2}x_0+\cdots+x_0^{n-1})(x-x_0)}{x-x_0}$
  $= \lim\limits_{x\to x_0} (x^{n-1} + x^{n-2}x_0 + \cdots + x_0^{n-1}) = x_0^{n-1} + x_0^{n-2}x_0 + \cdots + x_0^{n-1} = nx_0^{n-1}.$

$f\colon y = \mathrm{e}^x,\ x\in R.$

- $[\mathrm{e}^x]' = \lim\limits_{h\to 0} \frac{\mathrm{e}^{x+h} - \mathrm{e}^x}{h} = \lim\limits_{h\to 0} \frac{\mathrm{e}^x\cdot(\mathrm{e}^h - 1)}{h} = \mathrm{e}^x \cdot \lim\limits_{h\to 0} \frac{\mathrm{e}^h - 1}{h} = \mathrm{e}^x\cdot 1 = \mathrm{e}^x.$

We use various formulas a rules in the practical calculation of derivatives.

$f',\ g'$ exist on $A\neq\emptyset,\ c\in R.\ \Rightarrow$

- $(cf)',\ (f\pm g)',\ (fg)'$ exist on A, $\left(\frac{f}{g}\right)'$ exists on $A_1 = \{x\in A; g(x)\neq 0\}$.

In addition:

- $(cf)'(x) = cf'(x).$
- $(f\pm g)'(x) = f'(x) \pm g'(x).$
- $(fg)'(x) = f'(x)g(x) + f(x)g'(x).$
- $\left[\frac{f}{g}\right]'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.$

We briefly write the previous formulas:

- $(cf)' = cf'.$ • $(f\pm g)' = f'\pm g'.$ • $(fg)' = f'g + fg'.$ • $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}.$

$f\colon\ y = \frac{x}{x-1},\ x\in R - \{1\}$, line $p\colon\ y = 2 - x$.

- The tangents to the graph f are parallel to p $d_1\colon\ y = -x,\ d_2\colon\ y = 4 - x.$

The tangent $d\colon\ y = f(x_0) + f'(x_0)\cdot(x - x_0)$ at the point x_0 has the direction $f'(x_0)$.

The line p has a guideline of -1. $\Rightarrow\ f'(x_0) = -1.$

- $f'(x) = \left(\frac{x}{x-1}\right)' = \frac{1\cdot(x-1) - x(1-0)}{(x-1)^2} = \frac{x-1-x}{(x-1)^2} = \frac{-1}{(x-1)^2},\ x\in R - \{1\}.$
- $f'(x_0) = \frac{-1}{(x_0-1)^2} = -1.\ \Rightarrow\ (x_0 - 1)^2 = 1.\ \Rightarrow\ x_0 = 0$ alebo $x_0 = 2.$

Two tangents points $D = \left[x_0; f_(x_0)\right]$ a two tangents d:

- $D_1 = [0;0],\ d_1\colon\ y = 0 - (x - 0) = -x.$
- $D_1 = [2;2],\ d_2\colon\ y = 2 - (x - 2) = 4 - x.$

Derivative of inverse function

f is continuous a strictly monotone on the interval $I \subset R$.
$x_0 \in I$ is the inner point.
$f'(x_0) \neq 0$ is finite.
$\Rightarrow$

- The inverse function f^{-1} has a derivative at the point $y_0 = f(x_0)$ a holds
$$[f^{-1}]'(y_0) = \frac{1}{f'(x_0)}\bigg|_{x_0=f^{-1}(y_0)} = \frac{1}{f'(f^{-1}(y_0))}.$$

- $[f^{-1}]'(y_0) = \lim\limits_{y\to y_0} \frac{f^{-1}(y)-f^{-1}(y_0)}{y-y_0} = \left[\begin{array}{r|l} \text{Subst. } y = f(x) & x \to x_0 \\ x = f^{-1}(y) & y \to y_0 \end{array}\right]$
$$= \lim_{x\to x_0} \frac{x-x_0}{f(x)-f(x_0)} = \frac{1}{\lim\limits_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}} = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}.$$

We can simply write:

- $[f^{-1}]'(y) = \frac{1}{f'(x)} = \frac{1}{f'(f^{-1}(y))}$, resp. • $\frac{df^{-1}(y)}{dy} = \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{\frac{df(x)}{dx}}$.

f: $y = e^x$, $x \in R$ is continuous a increasing, $f'(x) = e^x \neq 0$ for $x \in R$.
f^{-1}: $x = \ln y$ for $y \in (0; \infty)$.

- $[\ln y]' = [f^{-1}]'(y) = \frac{1}{f'(x)} = \frac{1}{[e^x]'} = \frac{1}{e^x} = \frac{1}{e^{\ln y}} = \frac{1}{y}$ for $y \in (0; \infty)$.

f: $y = \sin x$, $x \in \left(-\frac{\pi}{2}; \frac{\pi}{2}\right)$ is continuous a increasing, $H(f) = (-1; 1)$.
$f'(x) = \cos x = \sqrt{1-\sin^2 x} > 0$ for $x \in \left(-\frac{\pi}{2}; \frac{\pi}{2}\right)$.

- $[\arcsin y]' = \frac{1}{[\sin x]'} = \frac{1}{\cos x} = \frac{1}{\sqrt{1-\sin^2 x}} = \frac{1}{\sqrt{1-[\sin \arcsin y]^2}} = \frac{1}{\sqrt{1-y^2}}$, $y \in (-1; 1)$.

Derivative of a function composition

$u = f(x)$, $y = g(u)$, $H(f) \subset D(g)$.
$x_0 \in D(f)$, $u_0 = f(x_0)$.
$f'(x_0)$, $g'(u_0)$ are finite.
$\Rightarrow$ • $[g(f(x_0))]' = g'(f(x_0)) \cdot f'(x_0)$
$= g'(u_0) \cdot f'(x_0)$.

We can simply write:

- $F'(x) = [g(f)]'(x) = g'(u) \cdot f'(x)$, resp. • $\frac{dF(x)}{dx} = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{dg(u)}{du} \cdot \frac{df(x)}{dx}$.

- $[\sin(\sin x)]' = \cos(\sin x)\cdot[\sin x]' = \cos(\sin x)\cdot\cos x,\ x\in R.$

- $[\sin(\sin(\sin x))]' = \cos(\sin(\sin x))\cdot[\sin(\sin x)]'$
  $= \cos(\sin(\sin x))\cdot\cos(\sin x)\cdot[\sin x]' = \cos(\sin(\sin x))\cdot\cos(\sin x)\cdot\cos x,\ x\in R.$

- $[a^x]' = \left[e^{\ln a^x}\right]' = \left[e^{x\ln a}\right]' = e^{x\ln a}\cdot[x\ln a]' = a^x\cdot\ln a,\ x\in R,\ a>0,\ a\neq 1.$

- $[x^a]' = \left[e^{\ln x^a}\right]' = \left[e^{a\ln x}\right]' = e^{a\ln x}\cdot[a\ln x]' = x^a\cdot\frac{a}{x} = ax^{a-1},\ x>0,\ a\in R.$

- $[x^x]' = \left[e^{\ln x^x}\right]' = \left[e^{x\ln x}\right]' = e^{x\ln x}\cdot[x\ln x]'$
  $= x^x\cdot\left[1\cdot\ln x + x\cdot\frac{1}{x}\right] = x^x\cdot[1+\ln x],\ x>0.$

- The expression $[\ln f(x_0)]' = \frac{f'(x_0)}{f(x_0)}$ is called **logarithmic derivative f at point x_0**.

Logarithmic derivative

$\left.\begin{array}{l} f(x_0)>0 \text{ for } x_0\in D(f).\\ f'(x_0) \text{ exists.}\end{array}\right\} \Rightarrow$ • $f'(x_0) = f(x_0)\cdot[\ln f(x_0)]'.$

Derivatives of basic elementary functions

Formula	Validity	Formula	Validity
$[c]' = 0,$	$x\in R,\ c\in R$	$[x]' = 1,$	$x\in R$
$[x^n]' = nx^{n-1},$	$x\in R,\ n\in N$	$[x^a]' = ax^{a-1},$	$x>0,\ a\in R$
$[e^x]' = e^x,$	$x\in R$	$[a^x]' = a^x\ln a,$	$x\in R,\ a>0$
$[\ln x]' = \frac{1}{x},$	$x>0$	$[\log_a x]' = \frac{1}{x\ln a},$	$x>0,\ a>0,\ a\neq 1$
$[\ln\lvert x\rvert]' = \frac{1}{x},$	$x\neq 0$	$[\log_a \lvert x\rvert]' = \frac{1}{x\ln a},$	$x\neq 0,\ a>0,\ a\neq 1$
$[\sin x]' = \cos x,$	$x\in R$	$[\cos x]' = -\sin x,$	$x\in R$
$[\tan x]' = \frac{1}{\cos^2 x},$	$x\neq\frac{(2k+1)\pi}{2},\ k\in Z$	$[\cot x]' = -\frac{1}{\sin^2 x},$	$x\neq k\pi,\ k\in Z$
$[\arcsin x]' = \frac{1}{\sqrt{1-x^2}},$	$x\in(-1;1)$	$[\arccos x]' = -\frac{1}{\sqrt{1-x^2}},$	$x\in(-1;1)$
$[\operatorname{arctg} x]' = \frac{1}{1+x^2},$	$x\in R$	$[\operatorname{arccotg} x]' = -\frac{1}{1+x^2},$	$x\in R$
$[\sinh x]' = \cosh x,$	$x\in R$	$[\cosh x]' = \sinh x,$	$x\in R$

Formula	Validity	Formula	Validity
$[\tanh x]' = \frac{1}{\cosh^2 x}$,	$x \in R$	$[\coth x]' = -\frac{1}{\sinh^2 x}$,	$x \neq 0$
$[\operatorname{argsinh} x]' = \frac{1}{\sqrt{x^2+1}}$,	$x \in R$	$[\operatorname{argcosh} x]' = \frac{1}{\sqrt{x^2-1}}$,	$x > 1$
$[\operatorname{argtgh} x]' = \frac{1}{1-x^2}$,	$x \in (-1;1)$	$[\operatorname{argcotgh} x]' = \frac{1}{1-x^2}$,	$x \in R - \langle -1;1\rangle$

The basis of successful derivative is the derivative of elementary functions. For practical needs, it is necessary to remember these patterns.

Differential function and higher order derivatives

Often we need to approximate the given function f (roughly express) another, simpler function of g so that their difference was $|f(x) - g(x)|$ as small as possible. Usually **local approximation** is enough for us in some neighborhood $O(x_0)$ point $x_0 \in D(f)$.

$y = f(x)$, $x \in D(f)$, bod $x_0 \in D(f)$, exists finity $f'(x_0)$.

- **Differential of function** f at point x_0, labeling $\mathrm{d}f(x_0, x-x_0)$, resp. $\mathrm{d}f(x_0, h)$

  is a linear function $\mathrm{d}f(x_0, x-x_0) = f'(x_0) \cdot (x-x_0)$, $x \in R$.

  Let $h = x-x_0$. $\Rightarrow$ • $\mathrm{d}f(x_0, x-x_0) = \mathrm{d}f(x_0, h) = f'(x_0) \cdot h$, $h \in R$.

f is**differentiable**:

- **at point** $x_0 \in D(f)$, if there exists $\mathrm{d}f(x_0, h)$, i.e. there is a finite $f'(x_0)$.
- **on set** $A \subset D(f)$, if $\mathrm{d}f(x_0, h)$ exists for all $x_0 \in A$.

$f\colon\ y = x$, $x \in R$, point $x_0 \in R$, $f'(x_0) = 1$.

• $\mathrm{d}f(x_0, h) = f'(x_0) \cdot h = 1 \cdot h = h$, $h \in R$, labeling $\mathrm{d}x$. $\Rightarrow$ • $\mathrm{d}f(x_0, h) = \mathrm{d}x$.

$f\colon\ y = f(x)$, $x \in R$, point $x_0 \in R$, $f'(x_0)$ is finite.

• $\mathrm{d}f(x_0, h) = f'(x_0) \cdot h = f'(x_0) \cdot \mathrm{d}x$, $h \in R$, labeling $\mathrm{d}f(x_0)$.

$\Rightarrow$ • $\mathrm{d}f(x_0, h) = \mathrm{d}f(x_0) = f'(x_0)\,\mathrm{d}x$, $f'(x_0) = \frac{\mathrm{d}f(x_0)}{\mathrm{d}x}$, resp. $f' = \frac{\mathrm{d}f}{\mathrm{d}x}$.

About the best local linear approximation

f is differentiable at $x_0 \in D(f)$.
h: $y = f(x_0) + c(x - x_0)$, $c \in R$, $c \neq f'(x_0)$.
g: $y = f(x_0) + f'(x_0)(x - x_0)$. $\Rightarrow$

- There exists a neighborhood $O(x_0)$ such that for all $x \in O(x_0)$, $x \neq x_0$ holds $|f(x) - g(x)| < |f(x) - h(x)|$.

- Approximation of f around $O(x_0)$ using the tangent at the point x_0

  g: $y = f(x_0) + f'(x_0)(x - x_0) = f(x_0)$, $x \in O(x_0)$

  is the best of all f approximations using a linear function.

$\sqrt[6]{1,06} \approx 1,01$. Exactly $\sqrt[6]{1,06} = 1,0097588$, calculation error $< 0,00025$.

Solution.
Let $f(x) = \sqrt[6]{x}$, $x > 0$, $x_0 = 1$.

$\Rightarrow$ • $f'(x) = [x^{1/6}]' = \frac{1}{6}x^{-5/6} = \frac{1}{6\sqrt[6]{x^5}}$, $x > 0$, $f'(x_0) = f'(1) = \frac{1}{6}$.

Let $O(1)$ also be that $1,06 \in O(1)$.

$\Rightarrow$ • $\sqrt[6]{x} = f(x) \approx f(1) + f'(1) \cdot (x - 1) = 1 + \frac{x-1}{6} = \frac{6+x-1}{6} = \frac{x+5}{6}$.

$\Rightarrow$ • $\sqrt[6]{1,06} = f(1,06) \approx \frac{1,06+5}{6} = \frac{6,06}{6} = 1,01$.

```
(%i8)  c:1.06$ f(x):=x^(1/6)$
       s:1$ f1(x):=diff(f(x),x,1)$ p(x):=f(s)+subst(s,x,f1(x))*(x-s)$ p(x);
       h(c):=print("c=",c,"          c^(1/6)=",'f(c),"=",float(f(c)),"approx",
       subst(c,x,float(p(x))))$ h(c)$
```

(%o6) $\frac{x-1}{6} + 1$

$c = 1.06 \quad c^{(1/6)} = f(1.06) = 1.009758794179192$ *approx* 1.01

- The f approximation only makes sense for x near the point x_0.

```
(%i18) h(0.9)$ h(1.1)$ h(1.2)$ h(1.5)$ h(2.0)$ h(4.0)$ h(10)$ h(16)$ h(32)$ h(64)$
```

$c = 0.9 \quad c^{(1/6)} = f(0.9) = 0.9825931938526898$ *approx* 0.9833333333333334
$c = 1.1 \quad c^{(1/6)} = f(1.1) = 1.016011867773387$ *approx* 1.016666666666667
$c = 1.2 \quad c^{(1/6)} = f(1.2) = 1.030853320886445$ *approx* 1.033333333333333
$c = 1.5 \quad c^{(1/6)} = f(1.5) = 1.069913193933663$ *approx* 1.083333333333333
$c = 2.0 \quad c^{(1/6)} = f(2.0) = 1.122462048309373$ *approx* 1.166666666666667
$c = 4.0 \quad c^{(1/6)} = f(4.0) = 1.259921049894873$ *approx* 1.5
$c = 10 \quad c^{(1/6)} = f(10) = 1.46779926762207$ *approx* 2.5
$c = 16 \quad c^{(1/6)} = f(16) = 1.587401051968199$ *approx* 3.5
$c = 32 \quad c^{(1/6)} = f(32) = 1.781797436280679$ *approx* 6.166666666666666
$c = 64 \quad c^{(1/6)} = f(64) = 2.0$ *approx* 11.5

$\sqrt[6]{1,06} \approx 1,01.$ Exactly $\sqrt[6]{1,06} = 1,0097588$, calculation error $< 0,00025$.

Another solution.
Let $f(x) = \sqrt[6]{x+1}$, $x > -1$, $x_0 = 0$.

$\Rightarrow$ • $f'(x) = [(x+1)^{1/6}]' = \frac{1}{6}(x+1)^{-5/6} = \frac{1}{6\sqrt[6]{(x+1)^5}}$, $x > 0$, $f'(x_0) = f'(0) = \frac{1}{6}$.

Let $O(0)$ also be that $0,06 \in O(0)$.

$\Rightarrow$ • $\sqrt[6]{x} = f(x) \approx f(0) + f'(0) \cdot x = 1 + \frac{x}{6} = \frac{x+6}{6}$.

$\Rightarrow$ • $\sqrt[6]{1,06} = f(0,06) \approx \frac{0,06+6}{6} = \frac{6,06}{6} = 1,01$.

```
(%i8) c:0.06$ f(x):=(x+1)^(1/6)$
      s:0$ f1(x):=diff(f(x),x,1)$ p(x):=f(s)+subst(s,x,f1(x))*(x-s)$ p(x);
      h(c):=print("c=",c,"          c^(1/6)=",'f(c),"=",float(f(c)),"approx",
      subst(c,x,float(p(x))))$ h(c)$
```

(%o6) $\frac{x}{6} + 1$

$c = 0.06 \quad (c+1)^{(1/6)} = f(1.06) = 1.009758794179192 \ approx \ 1.01$

$y = f(x)$, $x \in D(f)$ has the derivative f' on the set $A_1 \subset D(f)$, $A_1 \neq \emptyset$.

- $f' = f^{(1)}$ is called **first order derivative (first derivative)** f on the set A_1.
- Derivative f' (if exists), i.e. $[f']' = f'' = f^{(2)}$ on $A_2 \subset A_1$, $A_2 \neq \emptyset$ is called **second order derivative (second derivative)** f on the set A_2.
- Derivative f'' (if exists), i.e. $[f'']' = f''' = f^{(3)}$ on $A_3 \subset A_2$, $A_3 \neq \emptyset$ is called **third order derivative (third derivative)** f on the set A_3.
- Derivative f''' (if exists), i.e. $[f''']' = f^{(4)}$ on $A_4 \subset A_3$, $A_4 \neq \emptyset$ is called **fourth order derivative (fourth derivative)** f on the set A_4.
- This is how we continue for $n = 5, 6, 7, \ldots$.
- Derivative $f^{(n-1)}$, $n \in N$ (if exists), i.e. $[f^{(n-1)}]' = f^{(n)}$ on $A_n \subset A_{n-1}$, $A_n \neq \emptyset$ is called **derivative of the n-th order (n-th derivative)** f on the set A_n.
- We specially define $f = f^{(0)}$ **zero order derivative (zero derivative)** f.

$f^{(n)}(x_0)$ for $x_0 \in A_n$ is called **derivative of the n-th order (n-th derivative) f at point x_0**.

$$f^{(n)}(x_0) = \lim_{x \to x_0} \frac{f^{(n-1)}(x) - f^{(n-1)}(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f^{(n-1)}(x_0+h) - f^{(n-1)}(x_0)}{h}$$

for $x_0 \in A_n$, $A_n \subset A_{n-1}$, $n \in N$.

- This means that the function $f^{(n-1)}$ must be defined in some neighborhood $O(x_0)$.

Calculating $f^{(n)}$, $n\in N$ can be very laborious in general, because we have to start with f'.

$y=x^k$, $x\in R$, $k\in N$.

$$[x^k]'=kx^{k-1},\quad [x^k]''=k(k-1)x^{k-2},\quad [x^k]'''=k(k-1)(k-2)x^{k-3},\ \ldots,$$
$$[x^k]^{(k-1)}=k(k-1)\cdots 2x,\quad [x^k]^{(k)}=k!,\quad [x^k]^{(k+1)}=0,\ \ldots$$

$$\Rightarrow\ \bullet\ [x^k]^{(n)}=\begin{cases} k(k-1)\cdots(k-n+1)x^{k-n},\ x\in R & \text{for } n\in N,\ n\le k.\\ 0,\ x\in R & \text{for } n\in N,\ n>k.\end{cases}$$

$y=\mathrm{e}^x$, $x\in R$. $\Rightarrow$ $\bullet$ $[\mathrm{e}^x]^{(n)}=\mathrm{e}^x$, $x\in R$ for $n\in N$.

$y=\sin x$, $x\in R$, $y=\cos x$, $x\in R$.

$$[\sin x]'=\cos x,\quad [\sin x]''=[\cos x]'=-\sin x,\quad [\sin x]'''=[\cos x]''=-\cos x,$$
$$[\sin x]^{(4)}=[\cos x]'''=\sin x,\quad [\sin x]^{(5)}=[\cos x]^{(4)}=\cos x,\ \ldots$$

$$\Rightarrow\ \bullet\ [\sin x]^{(n)}=[\sin x]^{(n+4)}=\begin{cases}(-1)^k\sin x,\ x\in R & \text{for } n=2k,\ k\in N.\\ (-1)^{k+1}\cos x,\ x\in R & \text{for } n=2k-1,\ k\in N.\end{cases}$$

$$\Rightarrow\ \bullet\ [\cos x]^{(n)}=[\cos x]^{(n+4)}=\begin{cases}(-1)^k\sin x,\ x\in R & \text{for } n=2k,\ k\in N.\\ (-1)^{k}\cos x,\ x\in R & \text{for } n=2k-1,\ k\in N.\end{cases}$$

Leibniz's formula

f, g have derivatives on the set A to the order $n\in N$ (inclusive). $\Rightarrow$

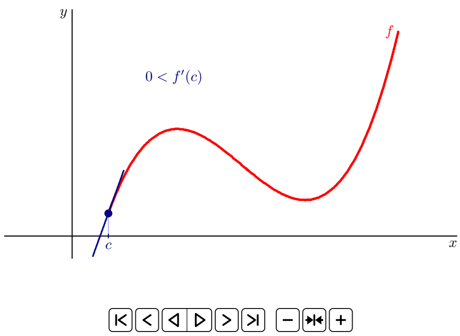
$$\bullet\ [fg]^{(n)}=\sum_{i=0}^{n}\binom{n}{i}f^{(n-i)}g^{(i)}=\binom{n}{0}f^{(n)}g^{(0)}+\binom{n}{1}f^{(n-1)}g^{(1)}+\cdots+\binom{n}{n}f^{(0)}g^{(n)}.$$

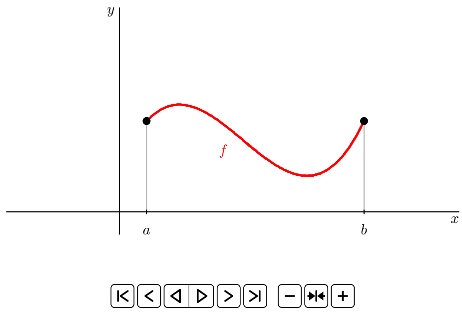
Applications of derivative

Theorems about the mean value of a function (Rolle a Lagrange) a l'Hospital's rule are among the most common applications of Derivative in practice.

Necessary condition for the existence of a local extreme

$$\left.\begin{array}{l} c\in D(f) \text{ is the inner point.}\\ f \text{ has a local extreme at } c.\\ f'(c) \text{ exists.}\end{array}\right\}\ \Rightarrow\ \bullet\ f'(c)=0.$$





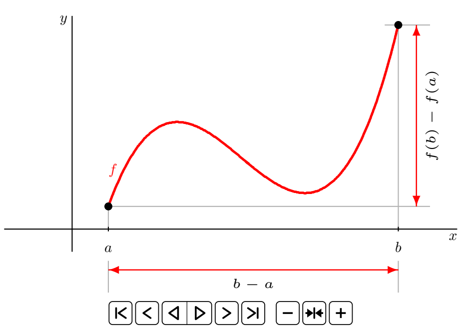
Necessary condition for the existence of a local extreme (left) and Rolle's theorem (right)

Rolle

f is continuous on $\langle a; b\rangle$.
$f(a) = f(b)$.
Exists $f'(x) \in R^*$ for all $x \in (a; b)$.
$\Rightarrow$ • Exists $c \in (a; b)$ so, that $f'(c) = 0$.

Lagrange (Mean value theorem)

f is continuous on $\langle a; b\rangle$.
Exists $f'(x) \in R^*$ for all $x \in (a; b)$.
$\Rightarrow$ • There exists $c \in (a; b)$ so, that $f'(c) = \frac{f(b)-f(a)}{b-a}$.



Lagrange's theorem

- $f'(c) = \frac{f(b)-f(a)}{b-a}$. $\Rightarrow$ • $f(b) - f(a) = f'(c) \cdot (b - a)$.

Let $b = a+h$, $h = b-a$, $h \in R$. $\Rightarrow$ $c = a + \theta(b-a) = a + \theta h$, $\theta \in (0; 1)$.

- $f(b) - f(a) = f(a + h) - f(a) = f'(a + \theta h) \cdot h$, $h \in R$, $\theta \in (0; 1)$.

For a sufficiently small h we can assume $f'(a+\theta h) \approx f'(a)$.

- $f(a+h) = f(a) + f'(a+\theta h)\cdot h \approx f(a) + f'(a)h = f(a) + \mathrm{d}f(a,h)$.

Rolle a Lagrange's theorem guarantee the existence of $c\in(a;b)$. However, we cannot use them to find such points, nor can we determine their number.

Indefinite expressions of type $\frac{0}{0}$, resp. $\frac{\infty}{\infty}$ sa often they are calculated using l'Hospital's rule.

L'Hospital's rule

For all $x\in O(a)$, $x\neq a$ exist $f'(x)$, $g'(x)$,
$a\in R^*$, exists $\lim\limits_{x\to a}\frac{f'(x)}{g'(x)} = b\in R^*$.
$\left\{\begin{array}{l}\lim\limits_{x\to a} f(x) = \pm\infty,\ \lim\limits_{x\to a} g(x) = \pm\infty\ [L'H\frac{\infty}{\infty}],\\ \text{resp. } \lim\limits_{x\to a} f(x) = 0,\ \lim\limits_{x\to a} g(x) = 0\ [L'H\frac{0}{0}].\end{array}\right.$

$\Rightarrow$ • Exists $\lim\limits_{x\to a}\frac{f(x)}{g(x)} = \lim\limits_{x\to a}\frac{f'(x)}{g'(x)} = b$.

$\lim\limits_{x\to 2}\frac{x^3-8}{x-2} = 12$.

- $f(x) = x^3-8$, $x\in R$, $g(x) = x-2$, $x\in R$.
- $O(2)$ can be chosen arbitrarily, e.g. $O(2) = R$.

$f'(x) = 3x^2$, $g'(x) = 1$ for $x\in R-\{2\}$.
$\lim\limits_{x\to 2}\frac{f'(x)}{g'(x)} = \lim\limits_{x\to 2}\frac{3x^2}{1} = 12$.
$\lim\limits_{x\to 2}(x^3-8) = \lim\limits_{x\to 2}(x-2) = 0$.

$\Rightarrow$ • $\lim\limits_{x\to 2}\frac{x^3-8}{x-2} = \lim\limits_{x\to 2}\frac{3x^2}{1} = 12$.

```
(%i9) f(x):=(x^3-8)/(x-2)$
      fc(x):=num(f(x))$ fc(x);
      fm(x):=denom(f(x))$ fm(x);
      'limit(f(x),x,2); 'limit(diff(fc(x),x,1)/diff(fm(x),x,1),x,2);
      limit(f(x),x,2); limit(diff(fc(x),x,1)/diff(fm(x),x,1),x,2);
```

(%o4) x^3-8

(%o5) $x-2$

(%o6) $\lim\limits_{x\to 2}\frac{x^3-8}{x-2}$

(%o7) $3\lim\limits_{x\to 2} x^2$

(%o8) 12

(%o9) 12

Without l'Hospital's rule:

- $\lim\limits_{x\to 2} \frac{x^3-8}{x-2} = \lim\limits_{x\to 2} \frac{(x-2)(x^2+2x+4)}{x-2} = \lim\limits_{x\to 2} (x^2+2x+4) = 4+4+4 = 12.$

- $\lim\limits_{x\to\infty} \frac{\ln x}{x} = [L'H\frac{\infty}{\infty}] = \lim\limits_{x\to\infty} \frac{[\ln x]'}{[x]'} = \lim\limits_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim\limits_{x\to\infty} \frac{1}{x} = \frac{1}{\infty} = 0.$

The preconditions of l'Hospital's rule are fulfilled:

- $[\ln x]' = \frac{1}{x}$, $[x]' = 1$ for $x \in (0;\infty)$.
- $\lim\limits_{x\to\infty} \frac{[\ln x]'}{[x]'} = \lim\limits_{x\to\infty} \frac{1}{x} = \frac{1}{\infty} = 0.$ • $\lim\limits_{x\to\infty} \ln x = \lim\limits_{x\to\infty} x = \infty.$

```
(%i4) f(x):=log(x)/x$ fc(x):=num(f(x))$ fm(x):=denom(f(x))$
      limit(diff(fc(x),x,1)/diff(fm(x),x,1),x,2);
(%o4) 1/2
```

- It is very important to verify all the assumptions of l'Hospital's rule.
- Validity of the assumption $\lim\limits_{x\to a} \frac{f'(x)}{g'(x)} = b \in R^*$ is verified continuously during the calculation of the limit.
- The reverse statement does not apply. The existence of $\lim\limits_{x\to a} \frac{f(x)}{g(x)}$ does not imply the existence of $\lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$.
- L'Hospital's rule can be used several times in a row:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = \lim_{x\to a} \frac{f''(x)}{g''(x)} = \cdots = \lim_{x\to a} \frac{f^{(k)}(x)}{g^{(k)}(x)},\ k \in N.$$

- $\lim\limits_{x\to 0} \frac{x-\sin x}{x^3} = [L'H\frac{0}{0}] = \lim\limits_{x\to 0} \frac{1-\cos x}{3x^2} = [L'H\frac{0}{0}] = \lim\limits_{x\to 0} \frac{0-(-\sin x)}{6x}$
$= \lim\limits_{x\to 0} \frac{\sin x}{6x} = [L'H\frac{0}{0}] = \lim\limits_{x\to 0} \frac{\cos x}{6} = \frac{1}{6}.$

The preconditions of l'Hospital's rule are fulfilled:

- For $O(0) = (-1;1)$, $x \in O(0)$, $x \neq 0$ there are corresponding derivatives and limits.

- $\lim\limits_{x\to\infty} \frac{e^x+e^{-x}}{e^x-e^{-x}} = [L'H\frac{\infty}{\infty}] = \lim\limits_{x\to\infty} \frac{e^x-e^{-x}}{e^x+e^{-x}} = [L'H\frac{\infty}{\infty}] = \lim\limits_{x\to\infty} \frac{e^x+e^{-x}}{e^x-e^{-x}} = \cdots$

We cannot apply L'Hospital's rule.

L'Hospital's rule can also be used to calculate other indefinite expressions. We must first convert them to type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by appropriate modifications.

Type $\pm\infty\cdot 0$: • $\lim\limits_{x\to a}[f(x)\cdot g(x)]$, kde $\lim\limits_{x\to a} f(x)=\pm\infty$, $\lim\limits_{x\to a} g(x)=0$.

- $\lim\limits_{x\to a}[f(x)g(x)]=\lim\limits_{x\to a}\frac{g(x)}{\frac{1}{f(x)}}$. $\Rightarrow$ Type $\frac{0}{0}$ $[L'H\frac{0}{0}]$.
- $\lim\limits_{x\to a}[f(x)g(x)]=\lim\limits_{x\to a}\frac{f(x)}{\frac{1}{g(x)}}$. $\Rightarrow$ Type $\frac{\infty}{\infty}$ $[L'H\frac{\infty}{\infty}]$.

Type $\infty-\infty$: • $\lim\limits_{x\to a}[f(x)-g(x)]$, where $\lim\limits_{x\to a} f(x)=\lim\limits_{x\to a} g(x)=\pm\infty$.

- $\lim\limits_{x\to a}[f(x)-g(x)]=\lim\limits_{x\to a}\left[\frac{f(x)g(x)}{g(x)}-\frac{f(x)g(x)}{f(x)}\right]=\lim\limits_{x\to a} f(x)g(x)\left[\frac{1}{g(x)}-\frac{1}{f(x)}\right]$. $\Rightarrow$ Type $\infty\cdot 0$.

Type ∞^0: • $\lim\limits_{x\to a}[f(x)]^{g(x)}$, where $\lim\limits_{x\to a} f(x)=\infty$, $\lim\limits_{x\to a} g(x)=0$.

- $\lim\limits_{x\to a}[f(x)]^{g(x)}=\lim\limits_{x\to a}\mathrm{e}^{\ln[f(x)]^{g(x)}}=\lim\limits_{x\to a}\mathrm{e}^{g(x)\cdot\ln f(x)}=\mathrm{e}^{\lim\limits_{x\to a} g(x)\cdot\ln f(x)}$. $\Rightarrow$ Type $0\cdot\infty$.

Type 0^0: • $\lim\limits_{x\to a}[f(x)]^{g(x)}$, where $\lim\limits_{x\to a} f(x)=\lim\limits_{x\to a} g(x)=0$.

- $\lim\limits_{x\to a}[f(x)]^{g(x)}=\lim\limits_{x\to a}\mathrm{e}^{\ln[f(x)]^{g(x)}}=\lim\limits_{x\to a}\mathrm{e}^{g(x)\cdot\ln f(x)}=\mathrm{e}^{\lim\limits_{x\to a} g(x)\cdot\ln f(x)}$. $\Rightarrow$ Type $0\cdot(-\infty)$.

Type $1^{\pm\infty}$: • $\lim\limits_{x\to a}[f(x)]^{g(x)}$, where $\lim\limits_{x\to a} f(x)=1$, $\lim\limits_{x\to a} g(x)=\pm\infty$.

- $\lim\limits_{x\to a}[f(x)]^{g(x)}=\lim\limits_{x\to a}\mathrm{e}^{\ln[f(x)]^{g(x)}}=\lim\limits_{x\to a}\mathrm{e}^{g(x)\cdot\ln f(x)}=\mathrm{e}^{\lim\limits_{x\to a} g(x)\cdot\ln f(x)}$. $\Rightarrow$ Type $\pm\infty\cdot 0$.

- $\lim\limits_{x\to 0^+}\sqrt[x]{x}=\lim\limits_{x\to 0^+}x^{1/x}=\mathrm{e}^{\lim\limits_{x\to 0^+}\ln x^{1/x}}=\mathrm{e}^{\lim\limits_{x\to 0^+}\frac{1}{x}\cdot\ln x}=\mathrm{e}^{\frac{1}{0^+}\cdot(-\infty)}$
$=\mathrm{e}^{\infty\cdot(-\infty)}=\mathrm{e}^{-\infty}=0.$

We did not apply L'Hospital's rule.

$y=f(x)$, $x\in D(f)$, point $x_0\in D(f)$, neighborhood $O(x_0)\subset D(f)$, $n\in N$.

There are finite derivatives $f'(x_0)$, $f''(x_0)$,..., $f^{(n)}(x_0)\in R$.

- **Taylor polynomial of degree n of function f with center at point x_0** is a function

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)\cdot(x-x_0)^k}{k!}$$

$$= f(x_0) + \frac{f'(x_0)\cdot(x-x_0)}{1!} + \frac{f''(x_0)\cdot(x-x_0)^2}{2!} + \cdots + \frac{f^{(n)}(x_0)\cdot(x-x_0)^n}{n!},\ x\in O(x_0).$$

If we denote $h = x-x_0$, $x = x_0+h$, $h\in O(0)$, then it has the form:

$$T_n(x_0+h) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)\cdot h^k}{k!} = f(x_0) + \frac{f'(x_0)\cdot h}{1!} + \frac{f''(x_0)\cdot h^2}{2!} + \cdots + \frac{f^{(n)}(x_0)\cdot h^n}{n!},\ h\in O(0).$$

- The Taylor polynomial $T_n(x)$ with center $x_0 = 0$ is called **Maclaurin polynomial**:

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)\cdot x^k}{k!} = f(0) + \frac{f'(0)\cdot x}{1!} + \frac{f''(0)\cdot x^2}{2!} + \cdots + \frac{f^{(n)}(0)\cdot x^n}{n!},\ x\in O(0).$$

- **The remainder of the Taylor polynomial** (degrees n) is called the difference

$$R_n(x) = f(x)-T_n(x) = \begin{cases} \frac{f^{(n+1)}(\xi)\cdot(x-x_0)^{n+1}}{(n+1)!},\ x\in O(x_0), & \text{Lagrangeov form,} \\ \frac{f^{(n+1)}(\xi)\cdot(x-x_0)\cdot(x-\xi)^n}{n!},\ x\in O(x_0), & \text{Cauchyho form,} \end{cases}$$

where $\xi = x_0 + \theta(x-x_0)$, $\theta\in(0;1)$.

The remainder $R_n(x)$ expresses the error of approximation f using the Taylor polynomial $T_n(x)$:

- The approximation has a local character around $O(x_0)$.
- The approximation is the best of all approximations using polynomials of degree n.

$f(x) = \sqrt[3]{1+x} = (x+1)^{\frac{1}{3}},\ x\in\langle -1;\infty),\ x_0 = 0.$ $\quad\Rightarrow\ f(0) = 1.$

- $f'(x) = \frac{1}{3}(x+1)^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{(x+1)^2}},\ x > -1.$ $\quad\Rightarrow\ f'(0) = \frac{1}{3}.$
- $f''(x) = -\frac{2}{3}\cdot\frac{1}{3}(x+1)^{-\frac{5}{3}} = \frac{-2}{9\sqrt[3]{(x+1)^5}},\ x > -1.$ $\quad\Rightarrow\ f''(0) = -\frac{2}{9}.$
- $f'''(x) = -\frac{5}{3}\cdot(-\frac{2}{9})\cdot(x+1)^{-\frac{8}{3}} = \frac{10}{27\sqrt[3]{(x+1)^8}},\ x > -1.$ $\quad\Rightarrow\ f'''(0) = \frac{10}{27}.$

$\Rightarrow$ • $T_3(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} = 1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{81},\ x\in O(0).$

- $\sqrt[3]{1+x} \approx \begin{cases} 1+\frac{x}{3}, & x\in O(0) \text{ with error } R_1(x). \\ 1+\frac{x}{3}-\frac{x^2}{9}, & x\in O(0) \text{ with error } R_2(x). \\ 1+\frac{x}{3}-\frac{x^2}{9}+\frac{5x^3}{81}, & x\in O(0) \text{ with error } R_3(x). \end{cases}$

Calculate the Taylor polynomial of the function $\sqrt{x^2+1}$. As we can see from the line (%i2), manual derivative is quite laborious.

```
(%i1) f(x):=sqrt(x^2+1)$
```

```
(%i2) print("f(x)=",f(x),", f'(x)=",diff(f(x),x),",
        f''(x)=",ratsimp(diff(f(x),x,2)),", f'''(x)=",ratsimp(diff(f(x),x,3)))$
```

$f(x) = \sqrt{x^2+1}$, $f'(x) = \frac{x}{\sqrt{x^2+1}}$, $f''(x) = \frac{\sqrt{x^2+1}}{x^4+2x^2+1}$, $f'''(x) = -\frac{3x\sqrt{x^2+1}}{x^6+3x^4+3x^2+1}$

For example, we see that the command `coeff` is dependent on the command `taylor`. Polynoma `tp1` is the ninth (practically eighth) degree, therefore the output command `coeff(tp1,x,10))` is the number 0. Polynoma `tp2` is the tenth degree a command output `coeff(tp2,x,10))` is real coefficient $c_{10} = 7/256$.

```
(%i3) tp1:taylor(f(x),x,0,9);
```

(tp1) $1 + \frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{16} - \frac{5x^8}{128} + \cdots$

```
(%i4) print("c_3=",coeff(tp1,x,3),",c_4=",coeff(tp1,x,4),",c_10=",coeff(tp1,x,10))$
```

$c\_3 = 0$, $c\_4 = -\frac{1}{8}$, $c\_10 = 0$

```
(%i5) tp2:taylor(f(x),x,0,10);
```

(tp2) $1 + \frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{16} - \frac{5x^8}{128} + \frac{7x^10}{256} + \cdots$

```
(%i6) print("c_3=",coeff(tp2,x,3),",c_4=",coeff(tp2,x,4),",c_10=",coeff(tp2,x,10))$
```

$c\_3 = 0$, $c\_4 = -\frac{1}{8}$, $c\_10 = \frac{7}{256}$

$f(x) = \ln x,\ x \in (0;\infty),\ x_0 = 1.$ $\Rightarrow$ $f(1) = 0.$

- $f'(x) = \frac{1}{x} = x^{-1},\ x > 0.$ $\Rightarrow$ $f'(1) = 1 = 0!.$
- $f''(x) = -\frac{1}{x^2} = -x^{-2},\ x > 0.$ $\Rightarrow$ $f''(1) = -1 = -1!.$
- $f'''(x) = 2\frac{1}{x^3} = 2x^{-3},\ x > 0.$ $\Rightarrow$ $f'''(1) = 2 \cdot 1 = 2!.$
- $f^{(4)}(x) = -3 \cdot 2\frac{1}{x^3} = -3 \cdot 2x^{-4},\ x > 0.$ $\Rightarrow$ $f^{(4)}(1) = 3 \cdot 2 \cdot 1 = -3!.$

  $\ldots$
- $f^{(k)}(x) = (-1)^{k-1}(k-1)!\frac{1}{x^{k-1}},\ x > 0,\ k \in N.$ $\Rightarrow$ $f^{(k)}(1) = (-1)^{k-1}(k-1)!.$

$\Rightarrow$ • $T_n(x) = 0 + \sum_{k=1}^{n} \frac{f^{(k)}(1)\cdot(x-1)^k}{k!} = \sum_{k=1}^{n} \frac{(-1)^{k-1}(k-1)!\cdot(x-1)^k}{k!}$

$$= \sum_{k=1}^{n} \frac{(-1)^{k-1}\cdot(x-1)^k}{k},\ x \in O(1).$$

```
(%i1) taylor(log(x),x,1,10);
```

(%o1) $x - 1 - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} - \frac{(x-1)^6}{6} + \frac{(x-1)^7}{7} - \frac{(x-1)^8}{8} + \frac{(x-1)^9}{9} - \frac{(x-1)^{10}}{10} + \cdots$

Sometimes it is more convenient to express $f(x) = \ln x$ in the form of a Maclaurin polynomial.

- $x = t+1.$ $\Rightarrow f(t) = \ln(t+1),\ t \in (-1;\infty),$

$$T_n(t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \frac{t^5}{5} + \cdots + \frac{(-1)^{n-1}t^n}{n} = \sum_{k=1}^{n} \frac{(-1)^{k-1}t^k}{k},\ t \in O(0).$$

(%i1) `taylor(log(x+1),x,0,10);`

(%o1) $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \frac{x^8}{8} + \frac{x^9}{9} - \frac{x^{10}}{10} + \cdots$

$f(x) = \mathrm{e}^x,\ x \in R.\ \Rightarrow$ Maclaurin polynomial of degree $n \in N$ has the form:

- $T_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} = \sum_{i=0}^{n} \frac{x^i}{i!},\ x \in R.$

$f(x) = \sin x,\ x \in R.\ \Rightarrow$ Maclaurin polynomial of degree $n \in N$ has the form:

- $T_{2k+1}(x) = 0 + \frac{x}{1!} + 0 + \frac{-x^3}{3!} + 0 + \frac{x^5}{5!} + 0 + \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} = \sum_{i=0}^{k} \frac{(-1)^i x^{2i+1}}{(2i+1)!},\ x \in R.$

$f(x) = \cos x,\ x \in R.\ \Rightarrow$ Maclaurin polynomial of degree $n \in N$ has the form:

- $T_{2k}(x) = 1 + 0 + \frac{-x^2}{2!} + 0 + \frac{x^4}{4!} + 0 + \cdots + \frac{(-1)^k x^{2k}}{(2k)!} = \sum_{i=0}^{k} \frac{(-1)^i x^{2i}}{(2i)!},\ x \in R.$

(%i1) `taylor(exp(x),x,0,10);`

(%o1) $1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + \frac{x^8}{40320} + \frac{x^9}{362880} + \frac{x^{10}}{3628800} + \cdots$

(%i2) `taylor(sin(x),x,0,10);`

(%o2) $x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \frac{x^9}{362880} + \cdots$

(%i3) `taylor(cos(x),x,0,10);`

(%o3) $1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320} - \frac{x^{10}}{3628800} + \cdots$

- The functions $y = \mathrm{e}^x$, $y = \sin x$, $y = \cos x$ can be approximated for each $x \in R$.
- We can achieve the required accuracy by increasing the degree of n sufficiently.

$f(x) = \mathrm{e}^{(x^2)},\ x \in R.$

Let $g(t) = \mathrm{e}^t,\ t \in R,\ t = x^2.\ \Rightarrow\ f(x) = \mathrm{e}^{(x^2)} = g(x^2) = g(t) = \mathrm{e}^t.$

For Maclaurin polynomial $P_n(t)$ function $g(t)$, $t \geq 0$

and Maclaurin polynomial $T_{2n}(x)$ function $f(x)$, $x \in R$ holds:

- $P_n(t) = \sum_{i=0}^{n} \frac{t^i}{i!} = \sum_{i=0}^{n} \frac{(x^2)^i}{i!} = \sum_{i=0}^{n} \frac{x^{2i}}{i!} = 1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots + \frac{x^{2n}}{n!} = T_{2n}(x).$

(%i1) `taylor(exp(x^2),x,0,10);`

(%o1) $1+x^2+\frac{x^4}{2}+\frac{x^6}{6}+\frac{x^8}{24}+\frac{x^{10}}{120}+\cdots$

(%i3) `subst(x^2,t,taylor(exp(t),t,0,5)); subst(x^2,t,taylor(exp(t),t,0,10));`

(%o2) $\frac{x^{10}}{120}+\frac{x^8}{24}+\frac{x^6}{6}+\frac{x^4}{2}+x^2+1$

(%o3) $\frac{x^{20}}{3628800}+\frac{x^{18}}{362880}+\frac{x^{16}}{40320}+\frac{x^{14}}{5040}+\frac{x^{12}}{720}+\frac{x^{10}}{120}+\frac{x^8}{24}+\frac{x^6}{6}+\frac{x^4}{2}+x^2+1$

At the end of this section we find a Maclaurin polynomial of degree 10 of the function $f(x)=\ln\frac{x^2+1}{x+1}$.

(%i1) `taylor(log((x^2+1)/(x+1)),x,0,10);`

(%o1) $-x+\frac{3x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\frac{x^5}{5}+\frac{x^6}{2}-\frac{x^7}{7}-\frac{x^8}{8}-\frac{x^9}{9}+\frac{3x^{10}}{10}+\cdots$

(%i3) `tp1(x):= taylor(log(x^2+1),x,0,10)-taylor(log(x+1),x,0,10)$ tp1(x);`

(%o3) $-x+\frac{3x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\frac{x^5}{5}+\frac{x^6}{2}-\frac{x^7}{7}-\frac{x^8}{8}-\frac{x^9}{9}+\frac{3x^{10}}{10}+\cdots$

(%i6) `tp2(x):=ratsimp(subst(x^2,t,taylor(log(t+1),t,0,5))-taylor(log(x+1),x,0,10))$`

`tp2(x); tp1(x)-tp2(x);`

(%o5) $\frac{756x^{10}-280x^9-315x^8-360x^7+1260x^6-504x^5-630x^4-840x^3+3780x^2-2520x}{2520}$

(%o6) $0+\cdots$

Investigation of behaviour of functions

An important part of the investigation of the behaviour of the function is the determination of intervals, on whose function is monotonic.

$I\subset R$ is the interval, f is continuous on I, for all $x\in I$ there exists $f'(x)\in R$. $\Rightarrow$

- f is on the interval I $\left\{\begin{array}{ll}\text{constant.} & \Leftrightarrow \bullet\ f'(x)=0\\ \text{increasing.} & \Leftrightarrow \bullet\ f'(x)>0\\ \text{nondecreasing.} & \Leftrightarrow \bullet\ f'(x)\geq 0\\ \text{decreasing.} & \Leftrightarrow \bullet\ f'(x)<0\\ \text{not increasing.} & \Leftrightarrow \bullet\ f'(x)\leq 0\end{array}\right\}$ for all $x\in I$.

Points where the continuous function f has local extrema, closely related to the intervals at which this function is strictly monotonic.

Necessary condition for the existence of a local extreme

$\left.\begin{array}{l}x_0\in D(f),\ f(x_0)\text{ is the local extreme.}\\ f'(x_0)\text{ exists.}\end{array}\right\}\Rightarrow \bullet\ f'(x_0)=0.$

- $f'(x_0) = 0$ does not guarantee a local extreme at x_0.
- The local extreme can also be the point at which the derivative does not exist.

When searching for local extrema of a function, we must:

- Investigate all points $x_0 \in D(f)$ for which $f'(x_0) = 0$ applies.
- Investigate all points $x_0 \in D(f)$ in which $f'(x_0)$ does not exist.

In addition, when searching for global extrema of a function, we must:

- Investigate the boundary points of $D(f)$.

The point $x_0 \in D(f)$ is called **stationary point of the function** f, if $f'(x_0) = 0$ exists.

Sufficient condition for local extreme existence

$x_0 \in D(f)$, $f'(x_0) = 0$. For all $x \in O(x_0)$ there exists $f'(x)$ and holds:

- $f'(x) > 0$ for $x < x_0$, $f'(x) < 0$ for $x > x_0$.
  $\Rightarrow$ • $f(x_0)$ is a strict local maximum.
- $f'(x) < 0$ for $x < x_0$, $f'(x) > 0$ for $x > x_0$.
  $\Rightarrow$ • $f(x_0)$ is a strict local minimum.
- $f'(x) < 0$ for $x \neq x_0$, resp. $f'(x) > 0$ for $x \neq x_0$.
  $\Rightarrow$ • $f(x_0)$ is not a local extreme.

$x_0 \in D(f)$, $f'(x_0) = 0$, $f''(x_0) \neq 0$ is finite. $\Rightarrow$

- $f''(x_0) < 0$. $\Rightarrow$ • $f(x_0)$ is a strict local maximum.
- $f''(x_0) > 0$. $\Rightarrow$ • $f(x_0)$ is a strict local minimum.

An important part of the investigation of the behaviour of the function is the determination of intervals, on whose function is convex or concave.

$I \subset R$ is the interval, for all $x \in I$ there exists $f'(x) \in R$. $\Rightarrow$

- f is on the interval I
  - convex. $\Leftrightarrow$ • f' is nondecreasing to I.
  - strictly convex. $\Leftrightarrow$ • f' is increasing on I.
  - concave. $\Leftrightarrow$ • f' is not increasing on I.
  - strictly concave. $\Leftrightarrow$ • f' is decreasing to I.

$I \subset R$ is the interval for all $x \in I$ exits $f''(x) \in R$. $\Rightarrow$

- f is on the interval I $\left\{ \begin{array}{lll} \text{strictly convex.} & \Leftrightarrow & \bullet\ f''(x) > 0 \\ \text{convex.} & \Leftrightarrow & \bullet\ f''(x) \geq 0 \\ \text{strictly concave.} & \Leftrightarrow & \bullet\ f''(x) < 0 \\ \text{concave.} & \Leftrightarrow & \bullet\ f''(x) \leq 0 \end{array} \right\}$ for all $x \in I$.

To investigate the convexity and concavity of the f function, we must:

- Investigate all points $x_0 \in D(f)$ for which $f''(x_0) = 0$ applies.
- Investigate all points of $x_0 \in D(f)$ where f is continuous a $f'(x_0)$ does not exist.

$\left. \begin{array}{l} x_0 \in D(f) \text{ is the inflection point of the function } f. \\ f''(x_0) \text{ exists.} \end{array} \right\} \Rightarrow \bullet\ f''(x_0) = 0.$

$x_0 \in D(f)$, $f'(x_0) \in R$. For all $x \in O(x_0)$ there exists $f''(x)$ and the following holds:

- $f''(x) > 0$ for $x < x_0$, $f''(x) < 0$ for $x > x_0$.
  $\Rightarrow$ • x_0 is the inflection point of the function f.
- $f''(x) < 0$ for $x < x_0$, $f''(x) > 0$ for $x > x_0$.
  $\Rightarrow$ • x_0 is the inflection point of the function f.
- $f''(x) < 0$ for $x \neq x_0$, resp. $f''(x) > 0$ for $x \neq x_0$.
  $\Rightarrow$ • x_0 is not an inflection point of function f.

$x_0 \in D(f)$, $f''(x_0) = 0$, $f'''(x_0) \neq 0$. $\Rightarrow$

- x_0 is the inflection point of the function f.

$x_0 \in D(f)$, $f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0$, $f^{(n)}(x_0) \neq 0$, $n \in N$.

$n = 2k-1$, $k \in N$ (odd). $\Rightarrow$
- $f(x_0)$ is not a local extreme.
- f is increasing at x_0 for $f^{(n)}(x_0) > 0$.
- f is decreasing at x_0 for $f^{(n)}(x_0) < 0$.

$n = 2k$, $k \in N$ (even). $\Rightarrow$
- $f(x_0)$ is the local extreme.
- $f(x_0)$ is strict minimum for $f^{(n)}(x_0) > 0$.
- $f(x_0)$ is strict maximum for $f^{(n)}(x_0) < 0$.

$x_0 \in D(f)$, $f'(x_0) \in R$, $f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0$, $f^{(n)}(x_0) \neq 0$, $n \in N$, $n \geq 2$.

$n = 2k+1$, $k \in N$ (odd). $\Rightarrow$
- x_0 is the inflection point of the function f.

$n = 2k$, $k \in N$ (even). $\Rightarrow$
- f is strictly convex at x_0 for $f^{(n)}(x_0) > 0$.
- f is strictly concave at x_0 for $f^{(n)}(x_0) < 0$.

Investigation of behaviour of functions

Investigating the behaviour of the function f means determining:

- Definition domain $D(f)$, points a intervals of continuity a discontinuity.
- Evenness, oddity, periodicity, resp. other special features.
- One-sided limits in points of discontinuity, in boundary points a in points $\pm\infty$.
- Zero points; intervals on which f is positive a negative.
- f', stationary points, local a global extrema; intervals at which f is increasing, decreasing, constant.
- f'', inflection points; intervals on which f is convex and concave.
- Asymptotes.
- $H(f)$ and sketch a function graph.

The graph will usually give us the most illustrative idea of the behaviour of the function. In its construction, we use all the data obtained. However, they are often insufficient, so we must supplement them with appropriately chosen functional values.

Behaviour of function $f(x) = \frac{8(x-2)}{x^2} = \frac{8x-16}{x^2}$.

```
(%i1) f(x):=(8*x-16)/x^2;
```
(%o1) $f(x) := \frac{8x-16}{x^2}$

- $D(f) = R - \{0\} = (-\infty; 0) \cup (0; \infty)$.

Using the command **denom** we find out when the denominator is zero.

```
(%i3) fm:denom(f(x));solve(fm=0,x);
```
(fmen) x^2
(%o3) $[x = 0]$

- f is not periodic, f is not even, f is not odd.
- f is continuous at intervals $(-\infty; 0)$, $(0; \infty)$, at point 0 is discontinuous.
- $\lim\limits_{x\to\pm\infty} f(x) = \lim\limits_{x\to\pm\infty} \frac{8x-16}{x^2} = \lim\limits_{x\to\pm\infty} \left(\frac{8}{x} - \frac{16}{x^2}\right) = \frac{8}{\pm\infty} - \frac{16}{\infty} = 0 - 0 = 0.$

```
(%i5) limit(f(x),x,minf);limit(f(x),x,inf);
```
(%o4) 0
(%o5) 0

- $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} \frac{8(x-2)}{x^2} = \frac{-16}{0^+} = -\infty$, $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \frac{8(x-2)}{x^2} = \frac{-16}{0^+} = -\infty$.

```
(%i7) limit(f(x),x,0,minus);limit(f(x),x,0,plus);
```
(%o6) $-\infty$
(%o7) $-\infty$

- Bod $x = 0$ je neodstrániteľný bod nespojitosti II. druhu. • The point $x = 0$ is an irremovable discontinuity of the II. type.
- $x = 0$ is an asymptote without a directive.
- $f(x) = \frac{8x-16}{x^2} = 0. \quad \Leftrightarrow \quad 8x - 16 = 0. \quad \Leftrightarrow \quad x = 2.$

Using the command **num** we find out when the numerator is zero.

```
(%i9) fcit:num(f(x));solve(fcit=0,x);
```
(fcit) $8x - 16$
(%o9) $[x = 2]$

- $x = 2$ je nulový bod f. $\Rightarrow \begin{cases} f(x) < 0 \text{ for } x \in (-\infty; 0), \\ f(x) < 0 \text{ for } x \in (0; 2), \\ f(x) > 0 \text{ for } x \in (2; \infty). \end{cases}$

$f(2) = 0$, f is not defined at $x = 0$.

$\Rightarrow$ The f function does not change the sign at intervals $(-\infty; 0)$, $(0; 2)$, $(2; \infty)$.

⇒ Just select any point in the given intervals a verify its value.

```
(%i13) f(2);f(-1);f(1);f(3);
(%o10) 0
(%o11) -24
(%o12) -8
(%o13) 8/9
```

- $f'(x) = \left[\frac{8x-16}{x^2}\right]' = \frac{8x^2-(8x-16)2x}{x^4} = \frac{32x-8x^2}{x^4} = \frac{32-8x}{x^3}$, $x \in R$, $x \neq 0$.

```
(%i15) f1(x):=diff(f(x),x,1)$ ratsimp(f1(x));
(%o15) -(8x-32)/x^3
```

- $f'(x) = \frac{32-8x}{x^3} = 0. \quad \Leftrightarrow \quad 32 - 8x = 0. \quad \Leftrightarrow \quad x = 4.$

```
(%i16) solve(f1(x)=0,x);
(%o16) [x = 4]
```

- f' is discontinuous at point 0.

```
(%i18) f1men:denom(ratsimp(f1(x)));solve(f1men=0,x);
(f1men) x^3
(%o18) [x = 0]
```

- $x = 4$ is the zero point f'. $\Rightarrow \begin{cases} f'(x) < 0, \ f \text{ is decreasing for } x \in (-\infty; 0), \\ f'(x) > 0, \ f \text{ is increasing for } x \in (0; 4), \\ f'(x) < 0, \ f \text{ is decreasing for } x \in (4; \infty). \end{cases}$

$f'(4) = 0$, f' is not defined at $x = 0$.

⇒ The f' function does not change the sign at intervals $(-\infty; 0)$, $(0; 4)$, $(4; \infty)$.

⇒ Just select any point in the given intervals a verify its value.

```
(%i22) subst(4,x,f1(x));subst(-1,x,f1(x));subst(1,x,f1(x));subst(5,x,f1(x));
(%o19) 0
(%o20) -40
(%o21) 24
(%o22) -8/125
```

- f has a local maximum at $x = 4$ and also a global maximum of $f(4) = 1$.

```
(%i23) f(4);
(%o23) 1
```

- f has no local or global minimum.
- $f''(x) = \left[\frac{32-8x}{x^3}\right]' = \frac{-8x^3-(32-8x)3x^2}{x^6} = \frac{16x^3-96x^2}{x^6} = \frac{16x-96}{x^4}$, $x \in R$, $x \neq 0$.

```
(%i25) f2(x):=diff(f(x),x,2)$ ratsimp(f2(x));
(%o25) (16x-96)/x^4
```

- $f''(x) = \frac{16x-96}{x^4} = 0. \;\Leftrightarrow\; 16x-96=0. \;\Leftrightarrow\; x=6.$

```
(%i26) solve(f2(x)=0,x);
(%o26) [x = 6]
```

- f'' is discontinuous at point 0.

```
(%i28) f2men:denom(ratsimp(f2(x)));solve(f2men=0,x);
(f2men) x^4
(%o28) [x = 0]
```

- $x=6$ is the zero point f''. $\Rightarrow \begin{cases} f''(x)<0,\ f \text{ is concave for } x\in(-\infty;0), \\ f''(x)<0,\ f \text{ is concave for } x\in(0;6), \\ f''(x)>0,\ f \text{ is convex for } x\in(6;\infty). \end{cases}$

$f'(6)=0$, f'' is not defined at $x=0$.

$\Rightarrow$ The f'' function does not change the sign at intervals $(-\infty;0)$, $(0;6)$, $(6;\infty)$.

$\Rightarrow$ Just select any point in the given intervals a verify its value.

```
(%i32) subst(6,x,f2(x));subst(-1,x,f2(x));subst(1,x,f2(x));subst(7,x,f2(x));
(%o29) 0
(%o30) -112
(%o31) -80
(%o32) 16/2401
```

- The point $x=6$ is the inflection point of the function f.

```
(%i33) f(6);
(%o33) 8/9
```

- $k = \lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \frac{8x-16}{x^3} = \lim_{x\to\pm\infty} \left(\frac{8}{x^2} - \frac{16}{x^3}\right) = 0-0=0.$
- $q = \lim_{x\to\pm\infty} [f(x)-kx] = \lim_{x\to\pm\infty} [f(x)-0\cdot x] = \lim_{x\to\pm\infty} f(x) = 0.$

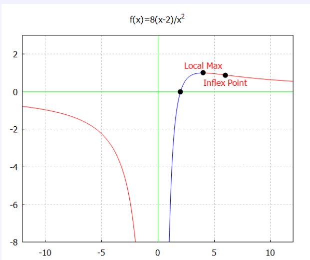
$\Rightarrow y = kx+q = 0.$

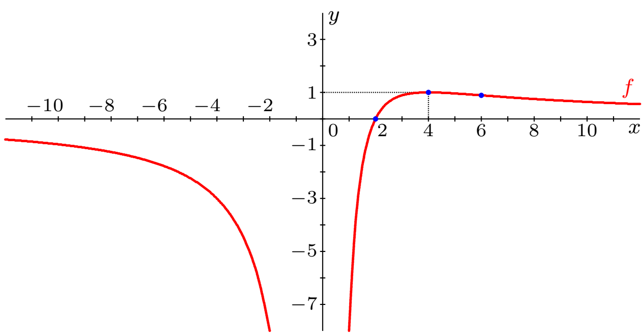
```
(%i35) km:limit(f(x)/x,x,minf);kp:limit(f(x)/x,x,inf);
(km) 0
(kp) 0
```

```
(%i37) qm:limit(f(x)-km*x,x,minf);qp:limit(f(x)-kp*x,x,inf);
(km)   0
(kp)   0
```

- $y = 0$ is an asymptote with a directive.
- $H(f) = (-\infty; 1\rangle$.

```
(%i38) draw2d(grid=true,xaxis=true,yaxis=true,xrange=[-12,12],yrange=[-8,3],
       title="f(x)=8(x-2)/x^2",color=blue,explicit(f(x),x,0,4),
       color=red,explicit(f(x),x,-12,0),explicit(f(x),x,4,12),
       label(["Inflex Point",6,f(6)-.4],["Local Max",4,f(4)+.4]),
       color=green,parametric(0,t,t,-8,3),parametric(t,0,t,-12,12),
       color=black,point_type=7,points([[4,f(4)],[6,f(6)],[2,f(2)]]))$
```





Graph of function $f(x) = \frac{8(x-2)}{x^2}$

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